Exact Order Reduction for State-Space Models of Multidimensional Systems

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(Abstract)

Completely different from the conventional one-dimensional (1-D) system, the minimal state-space realization of multidimensional systems is an extremely difficult problem. Thus, the existing realization processes often result in (non-minimal) models with relatively high orders. However, these high-order models will cause difficulties with respect to complexity and efficiency in system analysis and synthesis. It is, therefore, of great importance to develop exact order reduction approaches, which can reduce a given state-space model of a high order to a corresponding low-order one without introducing any approximation error or changing the input-out relation.

Inspired by the fact that the Popov-Belevitch-Hautus (PBH) tests characterize the reducibility of 1-D systems in terms of eigenvalues and eigenvectors, this thesis is devoted to studying the exact order reduction of n-D state-space models from the point of view of eigenvalues and eigenvectors. The key idea is to introduce the notion of common eigenvectors, by which we can successfully deal with multiple eigenvalues of n-D state-space models simultaneously and then establish a new approach to tackle this long-standing open problem. The main results and contributions are concerned with the following four aspects:

- First, a preliminary attempt is made for the n-D Roesser model based on a single eigenvalue, which only focuses on one sub-matrix corresponding one variable.
- Second, it is shown that by using the notions of multiple eigenvalues and constrained common eigenvector for multiple matrices, we can derive new reducibility conditions and the corresponding reduction procedures for the F-M model and the Roesser model, respectively. It will be clarified that this common eigenvector approach is applicable to a more general class of n-D state-space models for which the existing approaches fail to reach further order reduction. A Gröbner basis approach is also proposed to compute such a constrained common eigenvector, which leads to an alternative reducibility condition.

- Then, the common eigenvector reduction approach is further generalized based on the notion of common invariant subspace of multiple matrices. Specifically, more general reducibility conditions and the corresponding reduction procedures are presented for the n-D state-space models, which includes the results based on common eigenvectors as a special case.

- Finally, it is shown that an n-D Roesser model, which cannot be reduced by the proposed reduction methods and the other existing methods in the literature, may become reducible again when it is transformed into another equivalent Roesser model. Sufficient conditions and the corresponding procedure to derive such equivalent Roesser models are presented.

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Abbreviations

LFR	Linear Fractional Representation
F-M	Fornasini-Marchesini
LFT	Linear Fractional Transformation
LSI	Linear Shift-Invariant
MIMO	Multiple-Input and Multiple-Output
MFD	Matrix Fraction Description
n-D	Multidimensional
PBH	Popov-Belevitch-Hautus
SISO	Single-Input and Single-Output

Notation

$\oplus$	direct sum
$\otimes$	Kronecker product
$\triangleq$	is defined by
j	the imaginary unit which satisfies $j^2 = -1$
$\boldsymbol{z}$	a collection of variables $z_1, \ldots, z_n$
$\boldsymbol{i}$, $\boldsymbol{k}$	$(i_1, \ldots, i_n)$, $(k_1, \ldots, k_n)$
$\lvert\boldsymbol{i}\rvert$	$i_1 + \ldots + i_n$
$\{\ \}$	set
$\subset$	is a subset of
$\in$	is an element of
$\mathbf{R}$	the field of real numbers
$\mathbf{C}$	the field of complex numbers
$\mathbf{F}$	either $\mathbf{R}$ or $\mathbf{C}$
$\mathbf{Z}_+$	the set of nonnegative integers $\{0, 1, 2, \ldots\}$
$\mathbf{R}[\boldsymbol{z}]$	the polynomial ring $\mathbf{R}$ in n variables $z_1, \ldots, z_n$
$\mathbf{R}(\boldsymbol{z})$	the field of rational functions over $\mathbf{R}$ in n variables $z_1, \ldots, z_n$
$\mathbf{R}^{p \times q}$	the set of $p \times q$ real matrices
$\mathbf{C}^{n \times m}$	the set of $n \times m$ complex matrices

$\mathbf{F}^{n\times m}$	either $\mathbf{R}^{n\times m}$ or $\mathbf{C}^{n\times m}$
$\mathbf{R}^n$	$\mathbf{R}^{n\times 1}$ (set of real column vectors)
$\mathbf{C}^n$	$\mathbf{C}^{n\times 1}$ (set of complex column vectors)
$\mathbf{F}^n$	either $\mathbf{R}^n$ or $\mathbf{C}^n$
$\mathbf{R}^{p\times q}[\boldsymbol{z}]$	a set of $p \times q$ matrices with entries in $\mathbf{R}[\boldsymbol{z}]$
$\mathbf{R}^{p\times q}(\boldsymbol{z})$	a set of $p \times q$ matrices with entries in $\mathbf{R}(\boldsymbol{z})$
I_n	the $n \times n$ identity matrix
I	an identity matrix whose dimension can be inferred from context
$\mathbf{0}_{n\times m}$	the $n \times m$ zero matrix
$\mathbf{0}$	an $n \times m$ zero matrix whose dimension can be inferred from context
$\{\boldsymbol{e}_i\}_{i=1}^n$	the standard orthonormal basis for $\mathbf{R}^n$
$\dim(A)$	the dimension of the matrix A
$\ker(A)$	the kernel of the matrix A
$\mathrm{im}(A)$	the image of the matrix A
$\det\ (A)$	the determinant of the matrix A
$\mathrm{rank}\ (A)$	the rank of the matrix A
A^{T}	the transpose of the matrix A
A^{H}	the conjugate transpose of the matrix A
A^{-1}	inverse of matrix A
$A(i,k)$	the entry lying on the intersection of the ith row and kth column of A
$A(:,k)$	the kth column of A
$A(i,:)$	the ith row of A
x_i	the ith entry of $\boldsymbol{x} \in \mathbf{F}^n$

$\mathrm{diag}(A_1, \ldots, A_n)$	a block diagonal matrix such that A_i is its ith diagonal element
$\square$	end of proof

List of Algorithms

Contents

Chapter 1

Introduction

In this introductory chapter, the background and motivation to exact order reduction for state-space models of multidimensional systems are discussed. Then, four main contributions are delineated. Finally, the organization of the thesis is presented.

1.1 Background and Motivation

Over the last four decades, multidimensional (n-D) systems received a lot of attention due to their significance in both theory and practical applications such as image processing [1, 2], automatic control [3, 4], circuit analysis [5, 6], the real-time implementation of distributed grid sensor networks [7–10]. The characteristic feature of n-D systems is the presence of more than one, i.e., $n > 1$, independent variables, which implies that the signals in these dynamic systems propagate in n different directions, whereas those in one-dimensional (1-D) systems spread along one same direction. This essential difference makes it extremely awkward to generalize the conventional 1-D system theory to n-D case and it has been recognized that even some fundamental concepts and notions in the conventional 1-D system theory do not exist their counterparts in n-D system theory, e.g., the minimality of the realization (see [11–13] for the details). A fundamental problem for the n-D filters or systems is to construct a certain kind of n-D state-space model, typically the Roesser state-space model or the Fornasini-Marchesini (F-M) (second) state-space model, from a given transfer function or transfer matrix [14–17]. Moreover, Roesser model realization is basically equivalent to representing a parameter-dependent matrix as a linear fractional representation (LFR) for robustness analysis and synthesis of uncertain systems. It is difficult in general to obtain a minimal realization for a given n-D ($n \geq 2$

) system [18–20]. However, to improve the accuracy and to reduce the computational effort for the analysis of n-D system and the LFR-based robust control techniques, it is of paramount importance to develop effective procedures to generate realizations with lowest possible orders [16, 17].

There are in general two ways to achieve this goal. One is to directly develop realization approaches that can generate a Roesser model or an F-M model from a given n-D transfer function or matrix with order as low as possible. And the other is to develop order reduction approaches that can further reduce, if possible, the order of a given or known n-D Roesser model or F-M model in an exact manner, i.e., reduce the order of a given state-space model without introducing any approximation error or changing the original input-out relation.

For establishing effective realization approaches, considerable efforts have been made and a series of significant results have been obtained (see, e.g., [3, 11, 17, 19, 21–23] and the references therein). These realization methods can be generally classified into three main categories: the object-oriented LFR realization approach [3, 4], the elementary operation approach [17, 19, 22], the direct-construct realization approach [11, 20, 24, 25]. The object-oriented LFR realization approach [18, 26] starts with generation of elementary LFR objects for the parameters or variables in the given transfer function matrix. Then, some basic formulas for the combination of LFRs are employed to finally generate an over realization[18]. It is conceptually simple and can be easily implemented by software implementation [3, 4]. However, the combination of the elementary LFRs requires a lot of matrix calculations and complicated permutations for grouping together and sorting lexicographically the variables, which usually brings considerable computational burden [4, 27]. The elementary operation approach, which was initially proposed by Galkowski [22], obtains a Roesser model realization by performing appropriate elementary operations on a polynomial matrix. To overcome the main difficulties encountered by Galkowskis approach which often leads to a singular state-space realization even when a standard or regular realization exists, a new elementary operation approach is proposed [17, 19]. The advantages of this new elementary operation approach are that it provides a method to deal with the coefficient-dependent or field-dependent property of the n-D realization problem, and can be easily implemented a computer program in, e.g., MATLAB or Maple. The direction-construction realization approach is to directly generate an overall realization by constructing some special polynomial matrices (or vectors). The approach given in [24, 25] requires constructing two polynomial vectors, say $\boldsymbol{w}$ and $\boldsymbol{z}$, from the n-D monomials appearing in the given n-D polynomial matrix such that $\boldsymbol{w} = \Delta \boldsymbol{z}$ with Δ being a diagonal uncertainty block structure to be found. The drawback of this approach is that it makes the order of the resultant realization unnecessarily high, and it does not involve the coefficient values of the given transfer matrix. In order to obtain a realization with lower order, an alternative direct-construct Roesser model realization approach has been proposed in [11, 20]. Since the approach in [11, 20] is to compute a so-called admissible n-D polynomial matrix Ψ instead of monomial matrix for the given transfer matrix expressed as $G(z_1, \ldots, z_n) = N_r(z_1, \ldots, z_n) D_r^{-1}(z_1, \ldots, z_n)$ such that $N_r(z_1, \ldots, z_n) = CZ\Phi$ and $\Phi D^{-1}(z_1, \ldots, z_n) = (I - AZ)^{-1}$ with Z in the same structure as Δ, it can produce much lower realization order than the method of [24, 25]. Also, a series of direct-construction realization method have been developed for the F-M model realization (see, e.g., [14, 16, 28]).

However, for the exact order reduction of n-D state-space model, though some preliminary results have been reported in the literature [15, 29–32], there still remain many insights and issues to be explored.

It is well known that, for the conventional 1-D systems, a state-space is minimal (not reducible) if and only if it is both controllable and observable. In the context of n-D systems, however, the reducibility problem becomes much more complicated. Completely different from the conventional 1-D counterpart, the n-D state-space models have more complex structures involving n different variables. In particular, different blocks or submatrices of the state matrix of the n-D Roesser model correspond to different variables, which must be treated very carefully in various scenarios. The complex nature of n-D systems also make the controllability and observability much more difficult, and different notions on controllability and observability of n-D systems have been introduced. However, these notions are not very satisfactory in the sense that a state-space model can be minimal without being controllable or observable and conversely a system can be controllable and observable without being minimal. In other words, the relationship between reducibility and controllability or observability has not yet been clearly clarified.

Lambrechts et al. have later shown in [26] a way to apply the 1-D Kalman decomposition to the order reduction of an n-D system. That is, by taking a certain one of the n variables as the main variable and viewing the given n-D Roesser state-space model as a 1-D model with respect to the chosen main variable, it is possible to apply the 1-D Kalman decomposition to this n-D system, and repeating this operation successively to each of the left n-1 variables one can finally obtain a reduced-order n-D Roesser model. However, the effectiveness of this method is rather limited as it cannot deal with all the n variables simultaneously [33].

More recently, some novel results have been obtained in [30, 31, 34, 35] by restricting the paradigm to the so-called non-commuting n-D systems, in which, e.g., $z_1 z_2$ is not equal to $z_2 z_1$, for variables z_1 and z_2. By introducing the notions of structured (or generalized) Gramians, structured controllability/reachability and observability, it has been clarified in [31, 34] that a given non-commuting n-D system is reducible if and only if there exists a singular structured Gramian. In principle, this approach can also be applied to a commuting system by fixing it as certain non-commuting system. However, it is easy to see that the non-commutativity is a rather strong restriction and this method cannot lead to satisfactory order reduction in general. Therefore, the essential difficulty for the reduction problem of n-D systems remains challenging and new approaches are highly desired.

Moreover, all these exact order reduction are restricted to the Roesser model. Since the F-M model has totally different structural properties, the methods developed for the Roesser model cannot be directly applied to the F-M model. Furthermore, although embedding of a Roesser model into an F-M model preserves the order of the model, the reverse embedding requires, in general, increasing the order of the model [36, 37], and this fact means that it is also difficult to achieve an order reduction by first transforming an F-M model to an equivalent Roesser model and then using any exact order reduction methods for the equivalent Roesser model.

For the conventional 1-D systems, the well-known PBH tests for the controllability and observability reveals the relationship among the eigenvalues, the eigenvectors and the reducibility of a given 1-D state-sapce model [38]. In other words, the reducibility of a 1-D system has a close relationship with eigenvalues, eigenvectors. For example, for the conventional 1-D system

$$x' = Ax + Bu, \tag{1.1a}$$

$$y = Cx + Du, \tag{1.1b}$$

where $x \in \mathbf{R}^r$, $u \in \mathbf{R}^q$ and $y \in \mathbf{R}^p$ are the state vector, the input vector and the output vector, respectively; $A \in \mathbf{R}^{r\times r}$, $B \in \mathbf{R}^{r\times q}$, $C \in \mathbf{R}^{p\times r}$ and $D \in \mathbf{R}^{p\times q}$, if the matrix $\begin{bmatrix}(A-\lambda I_r)\\ C\end{bmatrix}$ is not full column rank for some eigenvalue $\lambda \in \mathbf{C}$ of the matrix A, or equivalently, the system matrix A has a right eigenvector $\boldsymbol{\omega}$ such that $C\boldsymbol{\omega} = \mathbf{0}$, then the given 1-D model is not observable and can be reduced [38].

Inspired by this fact, this thesis is devoted to studying the exact order reduction of n-D state-space models from the point of view of eigenvalues and eigenvectors without involving the difficulties for n-D controllability and observability. The key idea is to introduce the notions of multiple eigenvalues and common eigenvector, by which we can successfully deal with multiple coefficient matrices related to n-D state-space models simultaneously and thus establish a new approach to tackle this long-standing open problem.

1.2 Main Contributions

We now detail the specific contributions of this thesis, splitting them into four parts.

1.2.1 Eigenvalue Trim Approach to Exact Order Reduction for the Roesser Model

An eigenvalue trim approach is proposed to exact order reduction for the Roesser model based on a single eigenvalue, which explores essential insights on the connection between the eigenvalues and the reducibility of the corresponding n-D Roesser model, and to establish a more effective approach to exact order reduction for n-D Roesser models.

In particular, a new notion of eigenvalue trim or co-trim for the n-D Roesser (state-space) model is first introduced, which reveals the internal connection between the eigenvalues of the system matrix and the reducibility of the considered Roesser model. Then, new reducibility conditions and the corresponding order reduction algorithms based on eigenvalue trim or co-trim are proposed for exact order reduction of a given n-D Roesser model, and it will be shown that this eigenvalue trim approach can be applied even to those systems for which the existing approaches cannot do any further order reduction. Furthermore, a new transformation for n-D Roesser models, by swapping certain rows and columns and interchanging certain entries that belong to different blocks corresponding to different variables, will be established, which can transform an n-D Roesser model whose order cannot be reduced any more by the proposed approach to another equivalent Roesser model with the same order so that this transformed Roesser model can still be reduced further.

1.2.2 Common Eigenvector Approach to Exact Order Reduction for State-space Models of Multidimensional Systems

A common eigenvector approach is proposed to exact order reduction for state-space models of multidimensional systems by using multiple eigenvalues and common eigenvectors, which overcome the limitation of the eigenvalue trim approach on eigenvalues of one-sub-matrix. Specifically, the notion of constrained common eigenvectors is introduced, for the first time, which provides insight into the relationship between reducibility and multiple eigenvalues. Based on this result, new sufficient reducibility conditions and the corresponding reduction procedure are developed for n-D F-M models and Roesser models, respectively. It will be shown that this common eigenvector approach is applicable to a larger class of Roesser models for which the existing approaches may not be applied to do further order reduction. A Gröbner basis approach is proposed to compute such a constrained common eigenvector, which also leads to an equivalent reducibility condition.

1.2.3 Common Invariant Subspace Approach to Exact Order Reduction for State-space Models of Multidimensional Systems

The common eigenvector approach is extended based on a more general notion of the common invariant subspace. Then, an innovative common invariant subspace approach is derived for exact order reduction an n-D state-space model and it is clarified that this approach can generate a minimal state-space model of an n-D system in the non-commutative setting, which is from a point of view different to the methods reported in the literature. Specifically, new sufficient reducibility conditions based on common invariant subspaces are developed for the F-M model and the Roesser model, respectively. It is shown that the common invariant subspace approach includes the common eigenvector approach as a special case. Based on these new reducibility conditions, new constructive reduction procedures are given for the F-M model and the Roesser model, respectively.

1.2.4 Further Exact Order Reduction

The exact order reduction for the n-D Roesser model is further studied based on equivalence relationship. In particular, two types of transformations are firstly established to obtain equivalent Roesser models. It turns out that applying these two equivalent transformations to a minimal n-D Roesser model in the non-commutative setting can change the non-commutative transfer matrix of this n-D Roesser model and then the transformed n-D Roesser model may be reduced again by applying the common invariant subspace approach. Based on this fact, a novel reduction procedure is presented, which repeatedly applies the common invariant subspace approach to generate a minimal Roesser model realization in the non-commutative setting and the two equivalent transformations to obtain another Roesser model with different non-commutative transfer function matrices, such that an n-D Roesser model with order as low as possible can be obtained.

1.3 Outline of the Thesis

The thesis consists of eight chapters and is organized as follows.

In order to make the thesis readable and self-contained, some fundamental mathematical preparation and notions are given in the next chapter,

Chapter 3 summarizes some basic concepts of the n-D systems and the existing results of exact order reduction for n-D systems.

In Chapter 4, an eigenvalue trim approach is proposed to exact order reduction of an n-D Roesser model based on a single eigenvalue.

In Chapter 5, a common eigenvector approach is presented to exact order reduction for n-D state-space models by using multiple eigenvalues and common eigenvectors

In Chapter 6, an innovative common invariant subspace approach is derived for exact order reduction an n-D state-space model.

Chapter 7 further studies the exact order reduction for an n-D Roesser model based on equivalence relationship.

Chapter 8 summarizes the main results of this thesis and puts forward some further potential possibilities and problems for future research.

Chapter 2

Mathematical Preliminaries

In this chapter, some fundamental mathematical preparation and notions are give to make the thesis readable and self-contained.

2.1 Vector Spaces

Scalar field

Underlying a vector space is its *filed*, or set of scalars [39]. For our purpose, that underlying fields is typically the real numbers $\mathbb{R}$ or the complex numbers $\mathbf{C}$, but it could be the rational functions over $\mathbf{R}$ in n variables $z_1, \ldots, z_n$ denoted by $\mathbf{R}(z)$ and the complex rational functions over $\mathbf{C}$ in n variables $z_1, \ldots, z_n$ denoted by $\mathbf{C}(z)$. When the field is unspecified, we denoted it by the symbol $\mathbf{F}$ for numbers and $\mathbf{F}(z)$ for rational functions. To qualify as a *field*, a set must be closed under two binary operations: "addition" and "multiplication" satisfying [39]:

- both operations must be associative and commutative, and each must have an identity element in the set;
- inverses must exist in the set for all elements under addition and for all elements except the additive identity under multiplication;
- multiplication must be distributive over addition.

Vector Space

A vector space V over a field $\mathbf{F}$ is a set V along with an addition on V and a scalar multiplication on V such that the following properties hold [40]:

commutativity: $\boldsymbol{u}+\boldsymbol{v}=\boldsymbol{v}+\boldsymbol{u}$ for all $\boldsymbol{u},\boldsymbol{v}\in V$;

associativity: $(\boldsymbol{u}+\boldsymbol{v})+\boldsymbol{w}=\boldsymbol{u}+(\boldsymbol{v}+\boldsymbol{w})$ and $(ab)\boldsymbol{v}=a(b\boldsymbol{v})$ for $\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}\in V$ and $a,b\in\mathbf{F}$;

additive identity: there exists an element $\mathbf{0}\in V$ such that $\boldsymbol{v}+\mathbf{0}=\boldsymbol{v}$ for all $\boldsymbol{v}\in V$;

additive inverse: for every $\boldsymbol{v}\in V$, there exists $\boldsymbol{w}\in V$ such that $\boldsymbol{v}+\boldsymbol{w}=0$

multiplicative identity: $1\boldsymbol{v}=\boldsymbol{v}$ for all $\boldsymbol{v}\in V$;

distributive properties: $a(\boldsymbol{u}+\boldsymbol{v})=a\boldsymbol{u}+a\boldsymbol{v}$ and $(a+b)\boldsymbol{u}=a\boldsymbol{u}+b\boldsymbol{u}$ for all $a,b\in\mathbf{F}$ and all $\boldsymbol{u},\boldsymbol{v}\in V$.

For a given field $\mathbf{F}$ and a given positive integer r, the set $\mathbf{F}^r$ of r-tuples with entries from $\mathbf{F}$ forms a vector space over $\mathbf{F}$ under entrywise addition in $\mathbf{F}^r$ [39]. Our convention is that elements of $\mathbf{F}^r$ are always presented as column vectors; we often call them r-vectors. The special cases $\mathbf{R}^r$ and $\mathbf{C}^r$ are basic vector space of this thesis; $\mathbf{R}^r$ is a real vector space, i.e., a vector space over the real field, while $\mathbf{C}^r$ is both a real vector space and a complex vector space, i.e., a vector space over the complex field [39].

Subspace, Span, and Linear Combinations

A subset U of V is called a *subspace* of V if U is also a vector space (using the same addition and scalar multiplication as on V) [40]. If S is a subset of a vector space V over a filed F, span S is the intersection of all subspaces of V that contain S. If S is nonempty, i.e. $S=\{\boldsymbol{v}_1,\ldots,\boldsymbol{v}_k\}$, then

$$\operatorname{span}(S)=\{a_1\boldsymbol{v}_1+a_k\boldsymbol{v}_k:\boldsymbol{v}_1,\ldots,\boldsymbol{v}_k\in S,a_1,\ldots,a_k\in\mathbf{F}\}. \tag{2.1}$$

A linear combination of vectors in a vector space V over a field $\mathbf{F}$ is any expression of the form $a_1\boldsymbol{v}_1+a_k\boldsymbol{v}_k$ in which k is a positive integer, $a_1,\ldots,a_k\in\mathbf{F}$, and $\boldsymbol{v}_1,\ldots,\boldsymbol{v}_k\in V$ [39]. Thus, the span of a nonempty subset S of V consists of all linear combinations of finitely many vectors in S. A linear combination $a_1\boldsymbol{v}_1+\ldots a_k\boldsymbol{v}_k$ is *trivial* if $a_1=\ldots=a_k=0$; otherwise, it is nontrivial.

Let S_1 and S_2 be subspaces of a vector space over a field $\mathbf{F}$. The subspace is

$$S_1+S_2=\operatorname{span}\{S_1\cup S_2\}=\{\boldsymbol{u}+\boldsymbol{v}:\boldsymbol{u}\in S_1,\boldsymbol{v}\in S_2\} \tag{2.2}$$

called the *sum* of S_1 and S_2 [39]. If $\{S_1 \cup S_2\} = \{\mathbf{0}\}$, we say that the sum of S_1 and S_2 is a direct sum and write it as $S_1 \oplus S_2$ [39]; every $\boldsymbol{z} \in S_1 \oplus S_2$ can be written as $\boldsymbol{z} = \boldsymbol{u} + \boldsymbol{v}$ with $\boldsymbol{u} \in S_1$ and $\boldsymbol{u} \in S_2$ in one and only one way.

Linear dependence and linear independence

A finite list of vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k$ in a vector space V over a field $\mathbf{F}$ is *linearly dependent* if and only if there are scalars $a_1 \ldots, a_k \in \mathbf{F}$, not all zeros, such that $a_1\boldsymbol{v}_1 + \ldots a_k\boldsymbol{v}_k = 0$ [39]. Thus, a list of vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k$ is linearly dependent if and only if some nontrivial linear combination of $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k$ is the zero vector. It is often convenient to say that "vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k$ are linearly dependent" instead of the more formal statement "the list of vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k$ is linearly dependent."

A finite list of vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k$ in a vector space V over a field $\mathbf{F}$ is *linearly independent* if it is not linearly dependent [39]. Again, it can be convenient to say that "$\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k$ are linearly independent" instead of "the list of vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k$ is linearly independent."

Basis

A linearly independent list of vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k$ in a vector space V whose span is V is a *basis* for V. Each element of V can be represented as a linearly combination of vectors in a basis in one way and only one way;

Other detailed properties as well as more systematical definition related vector space can be found in, e.g., [39, 40].

2.2 Matrix

A *matrix* is an $m \times n$ (m-by-n) array of scalars from a field $\mathbf{F}$ [39]. If $m = n$, the matrix is said to be square. The set of all $m \times n$ matrices over $\mathbf{F}$ is denoted by $F^{m \times n}$. The vector space $F^{m \times 1}$ and F^m are identical. A *submatrix* of a given matrix is a rectangular array lying in specified subsets of the rows and columns of a given matrix.

Range Space and Null Space

The *range* of $A \in \mathbf{F}^{m \times n}$, denoted by range A, is defined by [41]:

$$\text{range}(A) \triangleq \{A\boldsymbol{x} : \boldsymbol{x} \in \mathbf{F}^n\} \tag{2.3}$$

It is known that (2.3) can be rewritten as

$$\mathrm{range}(A) = \mathrm{span}\{\sum_{k=1}^{n} x_k A(:,k)\}, \tag{2.4}$$

where x_k is the k-th entry of the vector $\boldsymbol{x}$. Thus, the range of A is also called its column space. The row space is $\{\boldsymbol{x}^{\mathrm{T}}A : \boldsymbol{x} \in \mathbf{F}^m\}$ [39].

The *null space* or *kernel space* is defined by

$$\mathrm{null}(A) \triangleq \{\boldsymbol{x} \in \mathbf{F}^m : A\boldsymbol{x} = \mathbf{0}\}. \tag{2.5}$$

The nullity of A, denoted by nullity(A), is the dimension of null(A); The *rank* of A, denoted by rank(A) is the dimension of range(A). These numbers are related by the *rank-nullity* theorem

$$\dim(\mathrm{rank}(A)) + \dim(\mathrm{null}(A)) = \mathrm{rank}(A) + \mathrm{nullity}(A) = n \tag{2.6}$$

for all $A \in F^{m\times n}$.

2.3 Eigenvalues, Eigenvectors, and Invariant Subspaces

2.3.1 Invariant Subspaces

Let $A : \mathbf{F}^n \to \mathbf{F}^n$ be a linear transformation, or say $A \in \mathbf{F}^{n\times n}$. A subspace $\mathcal{M} \in \mathbf{F}^n$ is called (right) invariant for the transformation A, or A-(right) invariant, if

$$A\boldsymbol{w} \in \mathcal{M} \tag{2.7}$$

for every $\boldsymbol{w} \in \mathcal{M}$ [42].

In other words, $\mathcal{M}$ is (right) invariant for A means that the image of $\mathcal{M}$ under A is contained in $\mathcal{M}$; $A\mathcal{M} \subset \mathcal{M}$. Trivial examples of (right) invariant subsapces are $\{\mathbf{0}\}$ and $\mathbf{C}^n$ [42]. Less trivial examples are the subspace

$$\mathrm{Ker}(A) = \{\boldsymbol{w} \in \mathbf{C}^n \ : \ A\boldsymbol{w} = \mathbf{0}\} \tag{2.8}$$

and

$$\mathrm{Im}(A) = \{A\boldsymbol{w} \ : \ \boldsymbol{w} \in \mathbf{C}^n\}. \tag{2.9}$$

A subspace W is called a common right invariant subspace of $A_1, \ldots, A_n$ if

$$\begin{aligned} A_1\boldsymbol{\omega} &\in W \\ &\vdots \\ A_n\boldsymbol{\omega} &\in W \end{aligned} \tag{2.10}$$

for every $\boldsymbol{\omega} \in W$. Similarly, a subspace W is called a common left invariant subspace of $A_1, \ldots, A_n$ if

$$\begin{aligned} \boldsymbol{\omega}^{\mathrm{T}} A_1 &\in W \\ &\vdots \\ \boldsymbol{\omega}^{\mathrm{T}} A_n &\in W \end{aligned} \tag{2.11}$$

for every $\boldsymbol{\omega} \in W$.

Some properties of the invariant subspace, which will be use full in this thesis, are stated as follows.

Lemma 2.1. *[43] A space W with a basis $\{\boldsymbol{w}_1, \ldots, \boldsymbol{w}_m\}$ is a right invariant under the linear transformation $A \in \mathbf{F}^{n\times n}$ if and only if*

$$A\boldsymbol{w}_i, \quad i \in \{1, \ldots, m\}, \tag{2.12}$$

is a linearly combination of the vectors $\boldsymbol{\omega}_1, \ldots, \boldsymbol{\omega}_m$.

Lemma 2.2. *A space W with a basis $\boldsymbol{\omega}_1, \ldots, \boldsymbol{\omega}_m$ is a common right invariant subspace matrices of matrices $A_1, \ldots, A_n$ if and only if*

$$A_i\boldsymbol{\omega}_k, \quad i \in \{1, \ldots, n\}, \quad k \in \{1, \ldots, m\}, \tag{2.13}$$

is a linearly combination of the vectors $\boldsymbol{\omega}_1, \ldots, \boldsymbol{\omega}_m$.

Other detailed properties about the invariant subspace can be found in, e.g., [42, 43].

2.3.2 Eigenvalues and Eigenvectors

Let $A \in \mathbf{C}^{n\times n}$. If a scalar λ and a nonzero vector $\boldsymbol{x}$ satisfy the equation

$$A\boldsymbol{x} = \lambda\boldsymbol{x}, \quad \boldsymbol{x} \in \mathbf{C}^n, \quad \boldsymbol{x} \neq \mathbf{0}, \quad \lambda \in \mathbf{C} \tag{2.14}$$

then λ is called an eigenvalue of A and $\boldsymbol{x}$ is called a (right) eigenvector of A associated with λ [39]. The pair λ, $\boldsymbol{x}$ is an eigenpair for A. Similarly, a vector $\boldsymbol{x}$ satisfying

$$\boldsymbol{x}^{\mathrm{T}} A = \lambda \boldsymbol{x}^{\mathrm{T}}, \quad \boldsymbol{x} \in \mathbf{C}^n, \quad \boldsymbol{x} \neq \mathbf{0}, \quad \lambda \in \mathbf{C} \tag{2.15}$$

is called a left eigenvector for A.

Remark 2.1. *For a given matrix $A \in \mathbf{C}^{n \times n}$ and let $\boldsymbol{W} = \mathrm{span}\{w\}$ with w being any eigenvector of A. It can be verified that the space $\boldsymbol{W}$ is an invariant subspace of A with dimension 1.*

Lemma 2.3. *[43] The span of a set of vectors that forms a chain of generalized eigenvectors for a matrix A corresponding to an eigenvlaue λ is an invariant subsapce for A.*

Chapter 3

Multidimensional Systems and Exact Order Reduction for State-Space Models

This chapter summarizes some basic concepts of the n-D systems and the existing results of exact order reduction for state-space models of n-D systems.

3.1 Multidimensional Signals and Systems

3.1.1 Multidimensional Signals

A *signal* is a physical quantity that can carry information, in general, about dynamic states and behaviors of a physical system. Mathematically, a signal can be represented by a function with certain variable. A multidimensional (n-D) signal is a signal which is represented by a function with n independent variables.

The independent variables and the amplitude of an n-D signal may be either continuous or discrete. For this reason, n-D signals are classified as continuous, discrete and digital signals. *Continuous signals* are signals whose independent variables are continuous and thus are represented by continuous variable functions. *Discrete signals* are those which possess discrete variables but continuous amplitudes and thus are characterized by sequences. *Digital signals* are those for which both independent variables and amplitudes are discrete. Additionally, signals whose independent variables and amplitudes are both continuous are sometimes referred to as *analog signals*. Since the main problems considered in this thesis are associated with n-D discrete systems, the n-D discrete signals which can be represented by sequences are in particular concerned.

A sequence u in n discrete variables (integer variables) $i_1, \ldots, i_n$ is formally expressed as

$$u = u(i_1, \ldots, i_n), \qquad -\infty < i_j < \infty, \quad j = 1, \ldots, n. \tag{3.1}$$

As in the 1-D case, some typical n-D discrete signals, such as unit-sample sequence, play important roles in n-D systems theory. The *unit-sample sequence* $\delta(i_1, \ldots, i_n)$, also often referred to as *unit impulse*, is defined as the sequence that is zero except at the origin, i.e.,

$$\delta(i_1, \ldots, i_n) = \begin{cases} 1, & i_1 = i_2 = \cdots = i_n = 0; \\ 0, & \text{otherwise.} \end{cases} \tag{3.2}$$

3.1.2 Linear, Shift-Invariant n-D Discrete Systems

A system which can be characterized by n independent variables is called n-D system. A single-input and single-output (SISO) n-D *discrete system* is mathematically defined as a transformation (or operator) that maps an input n-D discrete signal $\boldsymbol{u}(i_1, \ldots, i_n)$ into an output n-D discrete signal $\boldsymbol{y}(i_1, \ldots, i_n)$. More generally, a multi-input and multi-output (MIMO) n-D discrete system can be viewed as a transformation (or operator mapping) several input n-D discrete signals into several output n-D discrete signals. Figure 3.1

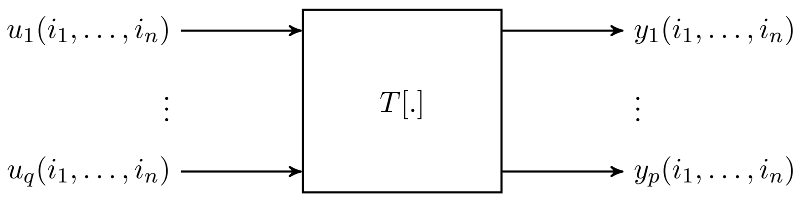

Figure 3.1: Representation of an MIMO n-D Discrete System.

illustrates a system that maps q inputs $u_1(i_1, \ldots, i_n), \ldots, u_q(i_1, \ldots, i_n)$ into p outputs $y_1(i_1, \ldots, i_n), \ldots, y_p(i_1, \ldots, i_n)$. The operator embodied in this system is represented by $T[\cdot]$, so by defining

$$\boldsymbol{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_q \end{bmatrix}, \qquad \boldsymbol{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_p \end{bmatrix}, \tag{3.3}$$

we may write

$$\boldsymbol{y}(i_1, \ldots, i_n) = T[\boldsymbol{u}(i_1, \ldots, i_n)]. \tag{3.4}$$

In general, the transformation $T[\cdot]$ may be rather complex. Due to different constraints imposed on $T[\cdot]$, classes of n-D discrete systems can be specified. Among them, an important class is the *linear shift-invariant systems* (LSI) which can be defined as follows.

Definition 3.1. *An n-D discrete system characterized by $T[\cdot]$ is said to be linear if and only if for any inputs $\boldsymbol{u}_1(i_1,\dots,i_n)$, $\boldsymbol{u}_2(i_1,\dots,i_n)$ and arbitrary nonzero constants c_1 and c_2,*

$$T[c_1\boldsymbol{u}_1(i_1,\dots,i_n)+c_2\boldsymbol{u}_2(i_1,\dots,i_n)] = c_1T[\boldsymbol{u}_1(i_1,\dots,i_n)]+c_2T[\boldsymbol{u}_2(i_1,\dots,i_n)]. \quad (3.5)$$

Let $\boldsymbol{y}(i_1,\dots,i_n)$ be the response to $\boldsymbol{u}(i_1,\dots,i_n)$, i.e., $\boldsymbol{y}(i_1,\dots,i_n) = T[\boldsymbol{u}(i_1,\dots,i_n)]$, the system is said to be shift-invariant if and only if for all $\boldsymbol{u}(i_1,\dots,i_n)$ and arbitrary integers $k_1,\dots,k_n$,

$$\boldsymbol{y}(i_1-k_1,\dots,i_n-k_n) = T[\boldsymbol{u}(i_1-k_1,\dots,i_n-k_n)]. \quad (3.6)$$

The system that satisfies both the above properties is then linear shift-invariant.

For an arbitrary n-D input vector sequence $\boldsymbol{u}(i_1,\dots,i_n)$, which can be expressed as

$$\boldsymbol{u}(i_1,\dots,i_n) \;=\; \sum_{k_1=-\infty}^{\infty}\cdots\sum_{k_n=-\infty}^{\infty}\boldsymbol{u}(k_1,\dots,k_n)\delta(i_1-k_1,\dots,i_n-k_n). \quad (3.7)$$

The output vector of an LSI system is

$$\boldsymbol{y}(i_1,\dots,i_n) = \sum_{k_1=-\infty}^{\infty}\cdots\sum_{k_n=-\infty}^{\infty} T[\boldsymbol{u}(k_1,\dots,k_n)\delta(i_1-k_1,\dots,i_n-k_n)]. \quad (3.8)$$

It has been proved that (3.8) can also be expressed in the form (see, e.g., [44] for the details)

$$\boldsymbol{y}(i_1,\dots,i_n) = \sum_{k_1=-\infty}^{\infty}\cdots\sum_{k_n=-\infty}^{\infty} H(i_1-k_1,\dots,i_n-k_n)\boldsymbol{u}(k_1,\dots,k_n). \quad (3.9)$$

This relation (3.9) is called the *convolution sum* of $H(i_1,\dots,i_n)$ with $\boldsymbol{u}(i_1,\dots,i_n)$ where $H(i_1,\dots,i_n)$ is the *impulse response* of the system with the ijth entry $h_{ij}(i_1,\dots,i_n)$ of $H(i_1,\dots,i_n)$ being the impulse response at the ith output port when the jth input signal is an n-D unit impulse $\delta(i_1,\dots,i_n)$ and all the other input signals are zero (see, e.g., [44]).

An n-D LSI system is said to be *causal* if its impulse response is zero outside the closed first quadrant of $\mathbb{R}^n$, i.e., $H(i_1,\dots,i_n)=\mathbf{0}$ if any $i_j<0$, $j\in\{1,2,\dots,n\}$ [45–47].

3.2 State-Space Representation of n-D Systems

As stated in Chapter 1, state-space representations play an important role in studying n-D systems and therefore different n-D state-space models have been intensively investigated (see [2, 48–53] and the references therein). The commonly used n-D state-space models are the Roesser model [2] and the Fornasini-Marchesini (F-M) (second) model [50].

3.2.1 The n-D Roesser State-Space Model

The Roesser state-space model for an n-D linear discrete system [2, 11, 15, 17, 29] is described by

$$\boldsymbol{x}'(i_1, \ldots, i_n) = A\boldsymbol{x}(i_1, \ldots, i_n) + B\boldsymbol{u}(i_1, \ldots, i_n), \tag{3.10a}$$

$$\boldsymbol{y}(i_1, \ldots, i_n) = C\boldsymbol{x}(i_1, \ldots, i_n) + D\boldsymbol{u}(i_1, \ldots, i_n), \tag{3.10b}$$

where $\boldsymbol{u}(i_1, \ldots, i_n) \in \mathbb{R}^q$ and $\boldsymbol{y}(i_1, \ldots, i_n) \in \mathbb{R}^p$ are the input and the output vectors, respectively; $\boldsymbol{x}(i_1, \ldots, i_n) \in \mathbb{R}^r$ is the local state vector in the form of

$$\boldsymbol{x}(i_1, \ldots, i_n) = \begin{bmatrix} \boldsymbol{x}_1(i_1, i_2, \ldots, i_n) \\ \vdots \\ \boldsymbol{x}_n(i_1, i_2, \ldots, i_n) \end{bmatrix}, \tag{3.11a}$$

$$\boldsymbol{x}'(i_1, \ldots, i_n) = \begin{bmatrix} \boldsymbol{x}_1(i_1 + 1, i_2, \ldots, i_n) \\ \vdots \\ \boldsymbol{x}_n(i_1, \ldots, i_{n-1}, i_n + 1) \end{bmatrix}, \tag{3.11b}$$

where $\boldsymbol{x}_k(i_1, i_2, \ldots, i_n) \in \mathbb{R}^{r_i}$ is the kth sub-state vector, $k = 1, \ldots, n$, $r = r_1 + \cdots + r_n$; and

$$A = \begin{bmatrix} A_{1,1} & A_{1,2} & \cdots & A_{1,n} \\ A_{2,1} & A_{2,2} & \ldots & A_{2,n} \\ \vdots & \vdots & \ldots & \vdots \\ A_{n,1} & A_{n,2} & \cdots & A_{n,n} \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_n \end{bmatrix}, \quad C = \begin{bmatrix} C_1 & C_2 & \cdots & C_n \end{bmatrix}, \tag{3.12}$$

and D are the real matrices of appropriate dimensions.

The dimension of the local state $\boldsymbol{x}(i_1, i_n, \ldots, i_n)$, or equivalently, the size of A, i.e., r, is called the *order* or *dimension* of the n-D Roesser model, while r_k is called the *partial order* or *dimension* w.r.t $\boldsymbol{x}_k(i_1, i_n, \ldots, i_n)$, for $k = 1, \ldots, n$, respectively. For explicitness, we also refer to the ordered n-tuple $(r_1, \ldots, r_n)$ as the order of a Roesser model. In what follows, $(A, B, C, D; \boldsymbol{r})$ with $\boldsymbol{r} = (r_1, \ldots, r_n)$ is used as a shorthand notation for the n-D Roesser model of the form (3.10).

To introduce the transfer matrix of the n-D F-M model (3.22), consider an n-D sequence, say $f(i_1, \ldots, i_n)$, $i_1 \geq 0$, ..., $i_n \geq 0$. Then, the n-D z transformation [54] of this sequence $f(i_1, \ldots, i_n)$, denoted by $Z\left[f(i_1, \ldots, i_n)\right]$, is defined as

$$Z\left[f(i_1, \ldots, i_n)\right] = \sum_{i_1=0}^{\infty} \cdots \sum_{i_n=0}^{\infty} f(i_1, \ldots, i_n) z_1^{i_1} \ldots z_n^{i_n}. \tag{3.13}$$

Let

$$Z \triangleq \mathrm{diag}\{z_1 I_{r_1}, \ldots, z_n I_{r_n}\}. \tag{3.14}$$

Note that such a block diagonal matrix Z is also often expressed by using direct sum $\oplus$ (see, e.g., [22, 39]), i.e., $Z = z_1 I_{r_1} \oplus \cdots \oplus z_n I_{r_n}$. Then applying the n-D transform to (3.10), we have

$$\mathcal{Z}\left[\boldsymbol{x}'(i_1, \ldots, i_n)\right] = A\mathcal{Z}\left[\boldsymbol{x}(i_1, \ldots, i_n)\right] + B\mathcal{Z}\left[\boldsymbol{u}(i_1, \ldots, i_n)\right], \tag{3.15a}$$

$$\mathcal{Z}\left[\boldsymbol{y}(i_1, \ldots, i_n)\right] = C\mathcal{Z}\left[\boldsymbol{x}(i_1, \ldots, i_n)\right] + D\mathcal{Z}\left[\boldsymbol{u}(i_1, \ldots, i_n)\right], \tag{3.15b}$$

where

$$\begin{aligned}
&\mathcal{Z}\left[\boldsymbol{x}'(i_1, \ldots, i_n)\right] = \sum_{i_1=0}^{\infty} \cdots \sum_{i_n=0}^{\infty} \boldsymbol{x}'(i_1, \ldots, i_n) z_1^{i_1} z_2^{i_2} \cdots z_n^{i_n} \\
&= \sum_{i_1=0}^{\infty} \cdots \sum_{i_n=0}^{\infty} \begin{bmatrix} \boldsymbol{x}_1(i_1+1, i_2, \ldots, i_n) \\ \vdots \\ \boldsymbol{x}_n(i_1, \ldots, i_{n-1}, i_n+1) \end{bmatrix} z_1^{i_1} z_2^{i_2} \cdots z_n^{i_n} \\
&= Z^{-1}\mathcal{Z}\left[\boldsymbol{x}(i_1, \ldots, i_n)\right] - \begin{bmatrix} z_1^{-1} \sum_{i_2=0}^{\infty} \cdots \sum_{i_n=0}^{\infty} \boldsymbol{x}_1(0, i_2, \ldots, i_n) z_1^0 z_2^{i_2} \cdots z_n^{i_n} \\ \vdots \\ z_n^{-1} \sum_{i_1=0}^{\infty} \cdots \sum_{i_{n-1}=0}^{\infty} \boldsymbol{x}_n(i_1, \ldots, i_{n-1}, 0) z_1^{i_1} z_2^{i_2} \cdots z_n^0 \end{bmatrix}.
\end{aligned} \tag{3.16}$$

Assuming the initial conditions $\boldsymbol{x}_k(i_1, \ldots, i_{k-1}, 0, i_{k+1}, \ldots, i_n)$, $i_t \in \mathbb{Z}_+$, $k, t = 1, \ldots, n$, are zeros, from (3.15) it can be shown that

$$\mathcal{Z}\left[\boldsymbol{y}(z_1, \ldots, z_n)\right] = H(z_1, \ldots, z_n)\mathcal{Z}\left[\boldsymbol{u}(z_1, \ldots, z_n)\right], \tag{3.17}$$

where

$$H(z_1, \ldots, z_n) = C(I_r - ZA)^{-1}ZB + D \tag{3.18}$$

is the transfer function matrix of the n-D Roesser model of (3.10).

Definition 3.2 (The n-D Roesser Model Realization)**.** *[17, 19, 27] Given an n-D rational transfer matrix $H(z_1, \ldots, z_n)$, find unknown matrices A, B, C, D and $\boldsymbol{r} = (r_1, r_2, \ldots, r_n)$ such that* (3.18) *holds.*

Remark 3.1. *It should be noted that the minimal realization of a general n-D rational transfer function matrix $H(z_1, \ldots, z_n)$, i.e., a realization with order $r = r_1 + \ldots + r_n$ is lowest among all the realizations $H(z_1, \ldots, z_n)$, is an extremely difficult problem [17, 19, 27]. Thus, in practice, it is desired to develop realization approaches to generate an n-D Roesser model with order as low as possible for a given n-D rational transfer function matrix.*

Such an n-D realization problem can also be used to obtain an LFR modeling for an uncertain system with a number of parameters, and vice versa [17, 19, 27]. The objective of LFR modelling is to represent $H(z_1, \ldots, z_n)$ as

$$H(z_1, \ldots, z_n) = \mathcal{F}_{\mathrm{u}}(M, \Delta) = D + C\Delta(I - A\Delta)^{-1}B, \tag{3.19}$$

with the partitioned matrix

$$M = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \in \mathbf{R}^{(r+p)\times(r+q)} \tag{3.20}$$

and with block-diagonal matrix

$$\Delta = \mathrm{diag}(z_1 I_{r_1}, \ldots, z_n I_{r_n}). \tag{3.21}$$

This LFR can be interpreted as the input-output mapping between u and y in Fig.3.2.

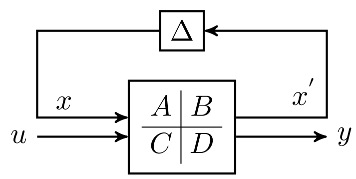

Figure 3.2: LFR realization of $H(z_1, \ldots, z_n)$.

3.2.2 The n-D F-M State-space Model

The F-M state-space model for an n-D linear discrete system [14, 16, 55, 56] is described by

$$\begin{aligned}
&\boldsymbol{x}(i_1+1, i_2+1, \ldots, i_n+1) \\
&\quad = A_1\boldsymbol{x}(i_1, i_2+1, \ldots, i_n+1) + \cdots + A_n\boldsymbol{x}(i_1+1, \ldots, i_{n-1}+1, i_n) \\
&\quad + B_1\boldsymbol{u}(i_1, i_2+1, \ldots, i_n+1) + \cdots + B_n\boldsymbol{u}(i_1+1, \ldots, i_{n-1}+1, i_n), \qquad (3.22a)
\end{aligned}$$

$$\boldsymbol{y}(i_1, \ldots, i_n) = C\boldsymbol{x}(i_1, \ldots, i_n) + D\boldsymbol{u}(i_1, \ldots, i_n), \qquad (3.22b)$$

where $\boldsymbol{x}(i_1, \ldots, i_n) \in \mathbf{R}^r$, $\boldsymbol{u}(i_1, \ldots, i_n) \in \mathbf{R}^q$, and $\boldsymbol{y}(i_1, \ldots, i_n) \in \mathbf{R}^p$ are the (local) s-tate vector, the input vector and the output vector, respectively; $A_1, \ldots, A_n \in \mathbf{R}^{r\times r}$, $B_1, \ldots, B_n \in \mathbf{R}^{r\times q}$, $C \in \mathbf{R}^{p\times r}$, $D \in \mathbf{R}^{p\times q}$; r is called the order of the n-D F-M model. In what follows, $(\boldsymbol{A}, \boldsymbol{B}, C, D; r)$ is used as a shorthand notation for the n-D F-M model of the form (3.22) with $\boldsymbol{A} = (A_1, \ldots, A_n)$ and $\boldsymbol{B} = (B_1, \ldots, B_n)$.

Remark 3.2. *The n-D F-M model of* (3.22) *plays an important role in some applications [8, 9]. For example, the F-M model was used to study the fault detection problem of 2-D and 3-D systems [57, 58]. The river pollution modeling issue was studied by making use of the 2-D F-M model in [59]. Moreover, the 3-D F-M model was applied to the real-time implementation of distributed grid sensor networks [8, 9].*

Applying the n-D z transformation to (3.22) with assumed zero boundary conditions gives, after routine algebraic manipulations,

$$Z[y(i_1, \ldots, i_n)] = H(z_1, \ldots, z_n)Z[u(i_1, \ldots, i_n)],$$

where $H(z_1, \ldots, z_n)$ is the transfer matrix given by

$$H(z_1, \ldots, z_n) = C\left(I_r - \sum_{i=1}^{n} z_iA_i\right)^{-1}\left(\sum_{i=1}^{n} z_iB_i\right) + D. \qquad (3.23)$$

Definition 3.3 (The n-D F-M Model Realization)**.** *[14, 16, 28] Given an n-D rational transfer function matrix $H(z_1, \ldots, z_n)$, find unknown matrices $A_1, \ldots, A_n, B_1, \ldots, B_n, C, D$ and an order r such that* (3.23) *holds.*

Remark 3.3. *It should be noted that the minimal F-M model realization of a general n-D rational transfer function matrix $H(z_1, \ldots, z_n)$, i.e., a realization with order r is lowest among all the realizations of $H(z_1, \ldots, z_n)$, is an extremely difficult problem [14, 16, 28]. Thus, in practice, it is desired to develop realization approaches to generate an n-D F-M model with order as low as possible for a given n-D rational transfer function matrix $H(z_1, \ldots, z_n)$.*

3.3 Exact Order Reduction Problem of State-Space Models

The exact order reduction for an n-D Roesser model and an n-D F-M model can be state as follows.

Problem 3.1 (Exact Order Reduction for an n-D Roesser model)**.** *For a given n-D Roesser model $(A, B, C, D; \boldsymbol{r})$ with $\boldsymbol{r} = (r_1, \ldots, r_n)$, find if possible another Roesser model $(\hat{A}, \hat{B}, \hat{C}, \hat{D}; \hat{\boldsymbol{r}})$ with $\boldsymbol{r} = (\hat{r}_1, \ldots, \hat{r}_n)$, such that*

$$\hat{C}\hat{Z}(I_{\hat{r}} - \hat{A}\hat{Z})^{-1}\hat{B} + \hat{D} = CZ(I_r - AZ)^{-1}B + D, \quad \hat{r} < r, \tag{3.24}$$

with

$$\hat{Z} = \mathrm{diag}\{z_1 I_{\hat{r}_1}, \ldots, z_n I_{\hat{r}_n}\} \tag{3.25}$$

and

$$\hat{r} = \hat{r}_1 + \ldots + \hat{r}_n. \tag{3.26}$$

Since substituting $z_1 = \ldots = z_n = 0$ into (3.24) yields $\hat{D} = D$, the exact order reduction problem for an n-D Roesser model can be simplified as follows.

Problem 3.2 (Exact Order Reduction for an n-D Roesser model)**.** *For a given n-D Roesser model $(A, B, C, D; \boldsymbol{r})$ with $\boldsymbol{r} = (r_1, \ldots, r_n)$, find if possible another Roesser model $(\hat{A}, \hat{B}, \hat{C}, \hat{D}; \hat{\boldsymbol{r}})$ with $\boldsymbol{r} = (\hat{r}_1, \ldots, \hat{r}_n)$ and $\hat{D} = D$, such that*

$$\hat{C}\hat{Z}(I_{\hat{r}} - \hat{A}\hat{Z})^{-1}\hat{B} = CZ(I_r - AZ)^{-1}B, \quad \hat{r} < r, \tag{3.27}$$

with $\hat{Z}$ of (3.25) and $\hat{r}$ of (3.26).

In the similar way, the exact order reduction problem for n-D F-M model can be simplified as follows.

Problem 3.3 (Exact Order Reduction for an n-D F-M model)**.** *For a given n-D F-M model $(\boldsymbol{A}, \boldsymbol{B}, C, D; r)$ with $\boldsymbol{A} = (A_1, \ldots, A_n)$ and $\boldsymbol{B} = (B_1, \ldots, B_n)$, find if possible another F-M model $(\hat{\boldsymbol{A}}, \hat{\boldsymbol{B}}, \hat{C}, \hat{D}; \hat{\boldsymbol{r}})$ with $\hat{\boldsymbol{A}} = (\hat{A}_1, \ldots, \hat{A}_n)$ and $\hat{\boldsymbol{B}} = (\hat{B}_1, \ldots, \hat{B}_n)$ such that*

$$\hat{C}\left(I_{\hat{r}} - \sum_{k=1}^{n} z_k \hat{A}_k\right)^{-1}\left(\sum_{k=1}^{n} z_k \hat{B}_k\right) + \hat{D} = C\left(I_r - \sum_{k=1}^{n} z_k A_k\right)^{-1}\left(\sum_{k=1}^{n} z_k B_k\right) + D, \tag{3.28a}$$

$$\hat{r} < r. \tag{3.28b}$$

Problem 3.4 (Exact Order Reduction for an$n$-D F-M model)**.** *For a given n-D F-M model $(\boldsymbol{A}, \boldsymbol{B}, C, D; r)$ with $\boldsymbol{A} = (A_1, \ldots, A_n)$ and $\boldsymbol{B} = (B_1, \ldots, B_n)$, find if possible another F-M model $(\hat{\boldsymbol{A}}, \hat{\boldsymbol{B}}, \hat{C}, \hat{D}; \hat{\boldsymbol{r}})$ with $\hat{\boldsymbol{A}} = (\hat{A}_1, \ldots, \hat{A}_n)$, $\hat{\boldsymbol{B}} = (\hat{B}_1, \ldots, \hat{B}_n)$ and $\hat{D} = D$ such that*

$$\hat{C}\left(I_{\hat{r}} - \sum_{k=1}^{n} z_k \hat{A}_k\right)^{-1}\left(\sum_{k=1}^{n} z_k \hat{B}_k\right) = C\left(I_r - \sum_{k=1}^{n} z_k A_k\right)^{-1}\left(\sum_{k=1}^{n} z_k B_k\right), \tag{3.29a}$$

$$\hat{r} < r\cdot \tag{3.29b}$$

3.4 Existing Results on Exact Order Reduction

For the conventional 1-D case, the notion of minimal realization is well suited and it is known that a realization is minimal if and only it is controllable and observable [38, 60]. However, in the multidimensional (n-D) setting, it has been shown that it is in general extremely difficult to derive minimal realization for n-D systems [29, 32]. Therefore, it is particular important to develop procedures which can reduce the order of state-space models generating by existing realization methods of a high order to a corresponding low-order one without introducing any approximation error or changing the input-out relation.

Up to now, great efforts have been paid to attack this difficult problem. In the earlier stage, much attention has been focused on the controllability and observability of n-D systems, resulting in various notions such as local, global, strong, and causal controllabilities and/or observabilities [61–67]). Unfortunately, these notions have no clear relevance to the reducibility of n-D systems, and it has been shown that an n-D state-space model can be minimal without being controllable or observable and conversely a system can be controllable and observable without being minimal (see, e.g., [67]).

Lambrechts et al. have later shown in [26] a way to apply the 1-D Kalman decomposition to the order reduction of n-D systems. That is, by taking a certain one of the n

variables as the main variable and viewing the given n-D Roesser state-space model as a 1-D one with respect to the chosen main variable, it is possible to apply the 1-D Kalman decomposition to this 1-D system, and repeating this operation successively to each of the left n-1 variables one can finally obtain a reduced-order n-D Roesser model. However, the effectiveness of this method is rather limited as it cannot deal with all the n variables simultaneously [33].

More recently, some novel results have been obtained in [30, 31, 34, 35], where it is shown that by restricting the paradigm to the so-called noncommuting n-D or NMD (noncommuting multidimensional) systems a nice n-D Kalman-like decomposition structure can be established which is directly relevant to the minimality, reachability and observability, and just includes the 1-D Kalman decomposition as a special case. Specifically, by introducing the notions of structured (or generalized) Gramians, structured (or NMD) controllability/reachability and observability, it has been clarified that a given noncommuting n-D system is reducible if and only if there exists a singular structured Gramian, or in other words, the existence of a singular structured Gramian implies that an equivalent n-D state-space model can be found which has a Kalman-like decomposition structure, and vice verse [31, 34]. It has also been shown that a given noncommuting n-D system is minimal if and only if it is structured controllable and observable, i.e., the associated structured controllability and observability matrices have, respectively, full ranks in the sense defined in [31, 34].

Though the results of [30, 31, 34, 35] are also applicable to the general commuting setting, i.e., the standard n-D systems, the reducibility and minimality conditions become only sufficient, which means that the essential difficulty for this problem still remains a challenge and thus some new approaches are desired.

Compared with the methods given in [4, 26, 30, 31], Sugie's method [24] is the first attempt to explore the further possible order reduction by utilizing the relationship and the permutations among the different blocks w.r.t. different variables. Since this kind of permutation for the rows and columns spans the different blocks, some additional conditions must be satisfied to keep the corresponding transfer (function) matrix unchanged. In order to make this method more applicable, Zerz has refined the idea initiated in [24] in a more theoretical way and relaxed the applicability conditions [23]. Yan el at. [32] study the exact order reduction problem related to the different blocks and have proposed a new order reduction approach, which transforms the order reduction problem to the problem of obtaining an objective matrix from a certain initial matrix by elementary operations so that the reduction is possible for a wider class of Roesser models than those shown in [23, 24]. It should be noted that a key condition required commonly by all the methods given in [23, 24, 32] is that at least one of the column (or row) blocks is not yet of full rank. To fully explore the exact order reduction problem for the n-D Roesser model, there still remain many insights and issues to be explored.

Furthermore, we would like to remark that all these methods on the exact order reduction of n-D systems are restricted to the Roesser model. Since exact order reduction approaches for the Roesser model have been established in the literature [29, 31, 32], a natural question is if the order reduction of a given F-M model can be implemented via transforming it to the corresponding Roesser model. As stated previously, this is in fact very difficult, since transforming an F-M model to a Roesser model requires, in general, increasing the dimensional of the local state space [36, 37], and there is no guarantee such that an F-M model can be obtained with order lower than that of the original one.

To see this more clearly, let us consider the simple 2-D F-M model given by

$$\begin{aligned} &A_1 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \\ &C = \begin{bmatrix} 1 & 1 \end{bmatrix}, \quad D = 0, \end{aligned} \tag{3.30}$$

which is a realization of $H(z_1, z_2) = \frac{-z_1 - 3\,z_2}{3\,z_1 + 2\,z_2 - 1}$. The 2-D F-M model of (3.30) can be embedded into the following 2-D Roesser model [36, 37]

$$\begin{bmatrix} x_1(i_1+1, i_2) \\ x_2(i_1, i_2+1) \end{bmatrix} = \begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{bmatrix} \begin{bmatrix} x_1(i_1, i_2) \\ x_2(i_1, i_2) \end{bmatrix} + \begin{bmatrix} B_{1,1} \\ B_{2,1} \end{bmatrix} u(i_1, i_2), \tag{3.31a}$$

$$y(i_1, i_2) = \begin{bmatrix} C_{1,1} & C_{1,2} \end{bmatrix} \begin{bmatrix} x_1(i_1, i_2) \\ x_2(i_1, i_2) \end{bmatrix} + D_{1,1} u(i_1, i_2), \tag{3.31b}$$

where $x_i(i_1, i_2)$ is the ith sub-state vector, $i \in \{1, 2\}$,

$$\begin{aligned} &\begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{bmatrix} = \begin{bmatrix} A_1 & A_1 \\ A_2 & A_2 \end{bmatrix}, \quad \begin{bmatrix} B_{1,1} \\ B_{2,1} \end{bmatrix} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \\ &\begin{bmatrix} C_{1,1} & C_{1,2} \end{bmatrix} = \begin{bmatrix} C & C \end{bmatrix}, \quad D_{1,1} = D, \end{aligned} \tag{3.32}$$

with order $r = 4$. Then, applying the reduction procedure of an n-D Roesser model given in [29] to (3.32) yields the 2-D Roesser model:

$$\begin{aligned} \tilde{A} &= \left[\begin{array}{c|c} 3 & 3 \\ \hline 2 & 2 \end{array}\right], \quad \tilde{B} = \left[\begin{array}{c} 1 \\ \hline 3 \end{array}\right], \\ \tilde{C} &= \left[\begin{array}{c|c} 1 & 1 \end{array}\right], \quad \tilde{D} = 0, \end{aligned} \tag{3.33}$$

with order $\tilde{r} = 2$, which is irreducible [13]. Finally, the embedding method given in [50] can transform it to the following 2-D F-M model:

$$\begin{aligned} &\tilde{A}_1 = \begin{bmatrix} 3 & 3 \\ 0 & 0 \end{bmatrix}, \quad \tilde{A}_2 = \begin{bmatrix} 0 & 0 \\ 2 & 2 \end{bmatrix}, \quad \tilde{B}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \tilde{B}_2 = \begin{bmatrix} 0 \\ 3 \end{bmatrix}, \\ &\tilde{C} = \begin{bmatrix} 1 & 1 \end{bmatrix}, \quad \tilde{D} = 0. \end{aligned} \tag{3.34}$$

It can be seen clearly that the order of the finally obtained F-M model of (3.34) is not less than that of the original F-M model of (3.30).

On the other hand, however, it can be easily checked that the 2-D F-M model:

$$\hat{A}_1 = 3, \quad \hat{A}_2 = 1, \quad \hat{B}_1 = 1, \quad \hat{B}_2 = -1, \quad \hat{C} = 3, \quad \hat{D} = 0 \tag{3.35}$$

and the given model (3.30) satisfy the relation (3.28), which shows that the order of (3.30) can be surely reduced to 1.

Based on the above discussion, we see that directly applying the results for the Roesser model to the F-M model is not feasible. Establishing different method aiming at the F-M model is worthwhile and this motivates our research.

Chapter 4

Eigenvalue Trim Approach to Exact Order Reduction for the Roesser Mode

In this Chapter, the exact order reduction of the Roesser model of n-D systems is treated by exploiting eigenvalues. Specifically, a new notion of eigenvalue trim or co-trim for an n-D Roesser (state-space) model is first introduced, which reveals the internal connection between the eigenvalues of the system matrix and the reducibility of the considered Roesser model. Then, new reducibility conditions and the corresponding order reduction algorithms based on eigenvalue trim or co-trim are proposed for exact order reduction of a given n-D Roesser model, and it will be shown that this eigenvalue trim approach can be applied even to those systems for which the existing approaches cannot do any further order reduction. Furthermore, a new transformation for n-D Roesser models, by swapping certain rows and columns and interchanging certain entries that belong to different blocks corresponding to different variables, will be established, which can transform an n-D Roesser model whose order cannot be reduced any more by the proposed approach to another equivalent Roesser model with the same order so that this transformed Roesser model can still be reduced further. Examples are given to illustrate the details as well as the effectiveness of the proposed approach.

This Chapter is organized as follows. In Section 4.1, the notion of the eigenvalue trim and its dual form of the eigenvalue co-trim will be introduced. Section 4.2, contains the main results: new conditions and the corresponding algorithms for order reduction of Roesser models. In Section 4.3, the transformation of changing an eigenvalue trim and eigenvalue co-trim n-D Roesser model to an equivalent but not eigenvalue trim or co-trim Roesser model is introduced. Then, comparisons with the order reduction methods of [23, 24, 29, 32] are discussed in Section 4.4. Finally, conclusions are given in 4.5.

4.1 Notion of the Eigenvalue Trim (Co-trim) Form

In this section, the notion on the eigenvalue trim and eigenvalue co-trim will be introduced for n-D Roesser models.

In order to explicitly see the differences to the existing results of [23], we first show the definition of trim and co-trim here.

Definition 4.1. *([23]) An n-D Roesser model given by $(A, B, C, D; \boldsymbol{r})$ is said to be trim if, with the partitions*

$$A = \begin{bmatrix} A_{1-} \\ \vdots \\ A_{n-} \end{bmatrix}, \quad A_{i-} \in \mathbf{R}^{r_i \times r}, \quad B = \begin{bmatrix} B_1 \\ \vdots \\ B_n \end{bmatrix}, \quad B_i \in \mathbf{R}^{r_i \times q}, \tag{4.1}$$

we have that for $1 \leq i \leq n$, each matrix $[A_{i-},\ B_i]$ has full row rank. Dually, $(A, B, C, D; \boldsymbol{r})$ is said to be co-trim if, with the partitions

$$\begin{aligned} A &= \begin{bmatrix} A_{-1} & \dots & A_{-n} \end{bmatrix}, \quad A_{-i} \in \mathbf{R}^{r \times r_i}, \\ C &= \begin{bmatrix} C_1 & \dots & C_n \end{bmatrix}, \quad C_i \in \mathbf{R}^{p \times r_i}, \end{aligned} \tag{4.2}$$

each $\begin{bmatrix} A_{-i} \\ C_i \end{bmatrix}$ has full column rank.

Let $\lambda_{i,1}, \dots, \lambda_{i,l_i}$ denote all the distinct eigenvalues of $A_{i,i}$ in (3.12) where $i \in \{1, \dots, n\}$, and l_i is the number of these eigenvalues. The notion of the eigenvalue trim and the eigenvalue co-trim can be defined as follows.

Definition 4.2. *An n-D Roesser model given by $(A, B, C, D; \boldsymbol{r})$ is said to be eigenvalue trim if with the partitions*

$$\begin{aligned} &A = \begin{bmatrix} A_{1-} \\ \vdots \\ A_{n-} \end{bmatrix}, \quad A_{i-} \in \mathbf{R}^{r_i \times r}, \quad B = \begin{bmatrix} B_1 \\ \vdots \\ B_n \end{bmatrix}, \quad B_i \in \mathbf{R}^{r_i \times q}, \\ &I_r = \begin{bmatrix} I_{1-} \\ \vdots \\ I_{n-} \end{bmatrix}, \quad I_{i-} \in \mathbf{R}^{r_i \times r}, \end{aligned} \tag{4.3}$$

we have that for $1 \leq i \leq n$ and $1 \leq t \leq l_i$, each matrix

$$
\begin{aligned}
\tilde{M}_{i-} &\triangleq \begin{bmatrix} (A_{i-} - \lambda_{i,t} I_{i-}) & B_i \end{bmatrix} \\
&= \begin{bmatrix} A_{i,1} & \dots & A_{i,i-1} & (A_{i,i} - \lambda_{i,t} I_{r_i}) & A_{i,i+1} & \dots & A_{i,n} & B_i \end{bmatrix}
\end{aligned} \tag{4.4}
$$

has full row rank. Dually, $(A, B, C, D; \boldsymbol{r})$ is said to be eigenvalue co-trim if with the partitions

$$
\begin{aligned}
A &= \begin{bmatrix} A_{-1} & \dots & A_{-n} \end{bmatrix}, \quad A_{-i} \in \mathbf{R}^{r \times r_i}, \\
C &= \begin{bmatrix} C_1 & \dots & C_n \end{bmatrix}, \quad C_i \in \mathbf{R}^{p \times r_i}, \\
I_r &= \begin{bmatrix} I_{-1} & \dots & I_{-n} \end{bmatrix}, \quad I_{-i} \in \mathbf{R}^{r \times r_i},
\end{aligned} \tag{4.5}
$$

we have that for $1 \leq i \leq n$ and $1 \leq t \leq l_i$, each matrix

$$
\begin{aligned}
\tilde{M}_{-i} &\triangleq \begin{bmatrix} A_{-i} - \lambda_{i,t} I_{-i} \\ C_i \end{bmatrix} \\
&= \begin{bmatrix} A_{1,i}^{\mathrm{T}} & \dots & A_{i-1,i}^{\mathrm{T}} & (A_{i,i}^{\mathrm{T}} - \lambda_{i,t} I_{r_i}) & A_{i+1,i}^{\mathrm{T}} & \dots & A_{n,i}^{\mathrm{T}} & C_i^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}
\end{aligned} \tag{4.6}
$$

has full column rank.

Remark 4.1. *It should be noted that eigenvalue trim and eigenvalue co-trim are dual, i.e., if a given Roesser model $(A, B, C, D; \boldsymbol{r})$ is eigenvalue trim, the Roesser model $(A^{\mathrm{T}}, C^{\mathrm{T}}, B^{\mathrm{T}}, D^{\mathrm{T}}; \boldsymbol{r})$ must be eigenvalue co-trim.*

As shown in the following lemma, eigenvalue trim (or eigenvalue co-trim) includes trim (or co-trim) just as a very special case.

Lemma 4.1. *An n-D Roesser model, which is eigenvalue trim (or eigenvalue co-trim), is always trim (or co-trim), however the reverse is not necessarily true.*

Proof. The result for eigenvalue co-trim is dual. Therefore, for simplicity, we only show the relationship between trim and eigenvalue trim.

The proof will be given by contradiction. Suppose that there is an n-D Roesser state-space model $(A, B, C, D; \boldsymbol{r})$ which is not trim but eigenvalue trim. Then, there is an index

$i \in \{1, \ldots, n\}$ such that

$$\begin{aligned} M_{i-} &\triangleq \left[\begin{array}{cc} A_{i-} & B_i \end{array} \right] \\ &= \left[\begin{array}{cccccccc} A_{i,1} & \ldots & A_{i,i-1} & A_{i,i} & A_{i,i+1} & \ldots & A_{i,n} & B_i \end{array} \right] \end{aligned} \tag{4.7}$$

has not full row rank and

$$\begin{aligned} \tilde{M}_{i-} &\triangleq \left[\begin{array}{cc} (A_{i-} - \lambda_{i,t} I_{i-}) & B_i \end{array} \right] \\ &= \left[\begin{array}{cccccccc} A_{i,1} & \ldots & A_{i,i-1} & (A_{i,i} - \lambda_{i,t} I_{r_i}) & A_{i,i+1} & \ldots & A_{i,n} & B_i \end{array} \right] \end{aligned} \tag{4.8}$$

is of full row rank for all distinct eigenvalues $\lambda_{i,t}$, $t = 1, \ldots, l_i$ of $A_{i,i}$.

Since the matrix M_{i-} in (4.7) does not have full row rank, there must exist a nonzero vector $\boldsymbol{\mu}$ such that

$$\boldsymbol{\mu} M_{i-} = \mathbf{0}. \tag{4.9}$$

This equation implies that

$$\begin{aligned} \boldsymbol{\mu} A_{i,i} &= \mathbf{0}\boldsymbol{\mu} \\ \boldsymbol{\mu} A_{i,v} &= \mathbf{0} \quad \text{for} \quad \text{all} \quad v \neq i \\ \boldsymbol{\mu} B_i &= \mathbf{0} \end{aligned} \tag{4.10}$$

which means that there is an eigenvalue $\lambda_{i,t} = 0$ of $A_{i,i}$ such that $\tilde{M}_{i-}$ in (4.8) is not of full row rank, which contradicts the assumption that the matrix $\tilde{M}_{i-}$ is not of full row rank for all the distinct eigenvalues of $A_{i,i}$. Hence, an eigenvalue trim Roesser model must also be trim.

It is easy to find an n-D Roesser model (e.g., (4.38) in Example 4.1 given later) that is trim but not eigenvalue trim. Hence, a trim n-D Roesser model is not necessarily eigenvalue trim.

Trim is equivalent to eigenvalue trim only in the case when all the eigenvalues are zero. □

Note that an n-D Roesser model is defined by real coefficient matrices A, B, C, D, and the complex eigenvalues of a real matrix come in pairs of complex conjugate numbers.

Then, we have the following lemma, which reveals that we only need to consider the real eigenvalues and the complex eigenvalues with positive imaginary parts when treating the order reduction problem.

Denote the imaginary unit by j, and let $\lambda_{i,1}, \ldots, \lambda_{i,\hat{l}_i}$ denote the distinct eigenvalues of $A_{i,i}$ with only real parts or with positive imaginary parts, where $i \in \{1, \ldots, n\}$, and $\hat{l}_i$ is the number of these distinct eigenvalues.

Lemma 4.2. *An n-D Roesser model given by $(A, B, C, D; \boldsymbol{r})$ is eigenvalue trim if and only if with the partitions (4.3) we have that for $1 \leq i \leq n$ and $1 \leq t \leq \hat{l}_i$, each matrix $\tilde{M}_{i-}$ in (4.4) is of full row rank. $(A, B, C, D; \boldsymbol{r})$ is eigenvalue co-trim if and only if with (4.5) we have that for $1 \leq i \leq n$ and $1 \leq t \leq \hat{l}_i$, each matrix $\tilde{M}_{-i}$ in (4.6) is of full column rank.*

Proof. The result for eigenvalue co-trim is dual. Therefore, for simplicity, we only show the proof for the eigenvalue trim.

By definition, if $(A, B, C, D, \boldsymbol{r})$ is eigenvalue trim, the matrix $\tilde{M}_{i-}$ in (4.4) is of full row rank for all distinct eigenvalues of $A_{i,i}$, and thus is of full row rank for the distinct eigenvalues with only real parts or with positive imaginary parts.

On the other hand, suppose that for all $1 \leq i \leq n$ if the matrix $\tilde{M}_{i-}$ in (4.4) is of full row rank for all distinct eigenvalues with only real parts or with positive imaginary parts, but (A, B, C, D) is not eigenvalue trim. Then, there is at least an eigenvalue $\bar{\lambda}_{i,t} \triangleq \alpha_i - j\beta_i$ with negative imaginary part such that $\tilde{M}_{i-}$ in (4.4) is not of full row rank. That is, there is a vector $\boldsymbol{\omega} \triangleq \boldsymbol{\mu} - j\boldsymbol{\nu}$ such that

$$\begin{aligned} &\boldsymbol{\omega} \left[\begin{array}{cccccccc} A_{i,1} & \ldots & A_{i,i-1} & (A_{i,i} - \bar{\lambda}_{i,t} I_{r_i} & A_{i,i+1}) & \ldots & A_{i,n} & B_i \end{array}\right] \\ =&(\boldsymbol{\mu} - j\boldsymbol{\nu}) \left[\begin{array}{cccccccc} A_{i,1} & \ldots & A_{i,i-1} & (A_{i,i} - (\alpha_i - j\beta_i) I_{r_i}) & A_{i,i+1} & \ldots & A_{i,n} & B_i \end{array}\right] = \mathbf{0} \end{aligned} \tag{4.11}$$

which gives that

$$(\boldsymbol{\mu} - j\boldsymbol{\nu})\, B_i = \mathbf{0}, \tag{4.12a}$$

$$(\boldsymbol{\mu} - j\boldsymbol{\nu})\, A_{i,i} = (\alpha_i \boldsymbol{\mu} - \beta_i \boldsymbol{\nu}) - j(\beta_i \boldsymbol{\mu} + \alpha_i \boldsymbol{\nu}), \tag{4.12b}$$

$$(\boldsymbol{\mu} - j\boldsymbol{\nu})\, A_{i,k} = \mathbf{0}, \quad k = 1, \ldots, i-1, i+1, \ldots, n, \tag{4.12c}$$

and then we have

$$\boldsymbol{\mu} B_i = \mathbf{0}, \quad \boldsymbol{\nu} B_i = \mathbf{0}, \tag{4.13a}$$

$$\boldsymbol{\mu} A_{i,i} = \alpha_i \boldsymbol{\mu} - \beta_i \boldsymbol{\nu}, \quad \boldsymbol{\nu} A_{i,i} = \beta_i \boldsymbol{\mu} + \alpha_i \boldsymbol{\nu}, \tag{4.13b}$$

$$\boldsymbol{\mu} A_{i,k} = \mathbf{0}, \quad \boldsymbol{\nu} A_{i,k} = \mathbf{0} \text{ for all } \quad k = 2, \ldots, n. \tag{4.13c}$$

From equation (4.13), we have

$$(\boldsymbol{\mu} + j\boldsymbol{\nu}) \begin{bmatrix} A_{i,1} & \ldots & A_{i,i-1} & (A_{i,i} - (\alpha_i + j\beta_i) I_{r_i}) & A_{i,i+1} & \ldots & A_{i,n} & B_i \end{bmatrix} = \mathbf{0}, \tag{4.14}$$

which means that $\tilde{M}_{i-}$ is not of full row rank for the complex eigenvalue $\lambda_{i,t} = \alpha_i + j\beta_i$ with positive imaginary part, which contradicts the assumption that the matrix $\tilde{M}_{i-}$ in (4.4) is of full row rank for all the complex eigenvalues with positive imaginary parts. □

4.2 Reducibility Based on Eigenvalue Trim

In this section, new conditions and the corresponding algorithms for order reduction of Roesser models will be developed.

Assume that λ_i is a complex eigenvalue of a matrix $A_{i,i}$, $i \in \{1, \ldots, n\}$. Then we have the following results.

Lemma 4.3. *For a given n-D Roesser model* $(A, B, C, D; \boldsymbol{r})$, *if the matrix* $\tilde{M}_{i-}$ *in (4.4) is not of full row rank for some complex eigenvalue* $\lambda_{i,t} = \lambda_i$ *of* $A_{i,i}$, $i \in \{1, \ldots, n\}$, *that is to say there exists a complex vector*

$$\boldsymbol{\omega} \triangleq \begin{bmatrix} \omega_1 & \ldots & \omega_{(k_0-1)} & 1 & \omega_{(k_0+1)} & \ldots & \boldsymbol{\omega}_{k_1} & \ldots & \omega_{r_i} \end{bmatrix}, \tag{4.15}$$

which can be expressed as

$$\begin{aligned} \boldsymbol{\omega} \triangleq \boldsymbol{\mu} + j\boldsymbol{\nu} \triangleq & \begin{bmatrix} \mu_1 & \cdots & \mu_{k_0-1} & 1 & \mu_{k_0+1} & \cdots & \mu_{k_1} \cdots \mu_{r_i} \end{bmatrix} \\ & + j \begin{bmatrix} \nu_1 & \cdots & \nu_{k_0-1} & 0 & \nu_{k_0+1} & \cdots & \nu_{k_1} \cdots \nu_{r_i} \end{bmatrix}, \end{aligned} \tag{4.16}$$

where j denotes imaginary unit, $\nu_{k_1} \neq 0$, *and* $\boldsymbol{\mu}$ *and* $\boldsymbol{\nu}$ *are real vectors such that*

$$\boldsymbol{\omega} \tilde{M}_{i-} = \mathbf{0}, \tag{4.17}$$

then one can construct a new Roesser model $(\hat{A}, \hat{B}, \hat{C}, D) \triangleq (LAR, LB, CR, D)$ *such that*

$$\hat{C}\hat{Z}(I_{\hat{r}} - \hat{A}\hat{Z})^{-1}\hat{B} = CZ(I_r - AZ)^{-1}B,$$

where $\hat{r}_i = r_i - 2$*;* $\hat{r}_k = r_k$ *for all* $k = 1, \ldots, i-1, i+1, \ldots, n$*;*

$$\hat{Z} = \mathrm{diag}\{z_1 I_{\hat{r}_1}, \ldots, z_n I_{\hat{r}_n}\}; \tag{4.18}$$

where

$$L \triangleq \mathrm{diag}\{I_{r_1}, \ldots, L_i, \ldots, I_{r_n}\}, \tag{4.19a}$$

$$R \triangleq \mathrm{diag}\{I_{r_1}, \ldots, R_i, \ldots, I_{r_n}\}, \tag{4.19b}$$

where L_i *is obtained from* $\tilde{L}_i$ *by deleting its* k_0th *and* k_1th *rows and* R_i *is obtained from* $\tilde{R}_i$ *by deleting its* k_0th *and* k_1th *columns with*

$$\tilde{L}_i \triangleq \begin{array}{c} \begin{array}{ccccccc} & & & \downarrow k_0 th & & \downarrow k_1 th & \phantom{\mu_{r_i}} \end{array} \\ \begin{bmatrix} 1 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & \cdots & 0 & \ldots & 0 \\ \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ \mu_1 & \mu_2 & \cdots & 1 & \cdots & \mu_{k_1} & \cdots & \mu_{r_i} \\ \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ \nu_1 & \nu_2 & \cdots & 0 & \cdots & \nu_{k_1} & \cdots & \nu_{r_i} \\ \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ 0 & 0 & \ldots & 0 & \ldots & 0 & \ldots & 1 \end{bmatrix} \end{array} \begin{array}{l} \\ \\ \\ \\ \leftarrow k_0 th \\ \\ \leftarrow k_1 th \\ \\ \\ \end{array}, \tag{4.20}$$

and $\tilde{R}_i \triangleq \tilde{L}_i^{-1}$.

Proof. Without loss of generality, we assume that $i = 1$, $k_0 = 1$ and $k_1 = 2$, since in the other case, the proof is similar and thus omitted. Note that

$$\tilde{L}_1 = \begin{bmatrix} \boldsymbol{\mu} \\ \boldsymbol{\nu} \\ L_1 \end{bmatrix} \triangleq \begin{bmatrix} 1 & \mu_2 & \check{\boldsymbol{\mu}} \\ 0 & \nu_2 & \check{\boldsymbol{\nu}} \\ 0 & 0 & I_{r_1-2} \end{bmatrix}, \tag{4.21a}$$

$$\tilde{R}_1 = \tilde{L}_1^{-1} = \begin{bmatrix} 1 & -\mu_2/\nu_2 & (\mu_2\check{\boldsymbol{\nu}} - \nu_2\check{\boldsymbol{\mu}})/\nu_2 \\ 0 & 1/\nu_2 & -\check{\boldsymbol{\nu}}/\nu_2 \\ 0 & 0 & I_{r_1-2} \end{bmatrix} \triangleq \begin{bmatrix} \hat{\boldsymbol{\mu}} & \hat{\boldsymbol{\nu}} & R_1 \end{bmatrix}. \tag{4.21b}$$

For the complex λ_1, one can express it as $\lambda_1 \triangleq \alpha_1 + j\beta_1$ where j denotes the imaginary unit and α_1 and β_1 are real numbers. From equation (4.17), we have

$$\boldsymbol{\omega} B_1 = \mathbf{0}, \tag{4.22a}$$

$$\boldsymbol{\omega}\, A_{1,1} = \lambda_1 \boldsymbol{\omega} = (\alpha_1 + j\beta_1)\,\boldsymbol{\omega}, \tag{4.22b}$$

$$\boldsymbol{\omega}\, A_{1,k} = \mathbf{0}, \quad \text{for} \quad k = 2, \ldots, n. \tag{4.22c}$$

Substituting equation (4.16) into (4.22) we obtain

$$(\boldsymbol{\mu} + j\boldsymbol{\nu})\, B_1 = \mathbf{0}, \tag{4.23a}$$

$$(\boldsymbol{\mu} + j\boldsymbol{\nu})\, A_{1,1} = (\alpha_1 \boldsymbol{\mu} - \beta_1 \boldsymbol{\nu}) + j(\beta_1 \boldsymbol{\mu} + \alpha_1 \boldsymbol{\nu}), \tag{4.23b}$$

$$(\boldsymbol{\mu} + j\boldsymbol{\nu})\, A_{1,k} = \mathbf{0}, \quad \text{for} \quad k = 2, \ldots, n, \tag{4.23c}$$

which gives that

$$\boldsymbol{\mu} B_1 = \mathbf{0}, \quad \boldsymbol{\nu} B_1 = \mathbf{0}, \tag{4.24a}$$

$$\boldsymbol{\mu} A_{1,1} = \alpha_1 \boldsymbol{\mu} - \beta_1 \boldsymbol{\nu}, \quad \boldsymbol{\nu} A_{1,1} = \beta_1 \boldsymbol{\mu} + \alpha_1 \boldsymbol{\nu}, \tag{4.24b}$$

$$\boldsymbol{\mu} A_{1,k} = \mathbf{0}, \quad \boldsymbol{\nu} A_{1,k} = \mathbf{0} \quad \text{for all} \quad k = 2, \ldots, n. \tag{4.24c}$$

From equation (4.21a) and (4.24c), we have

$$\begin{aligned} \tilde{L}_1 A_{1,k} &= \begin{bmatrix} \boldsymbol{\mu} \\ \boldsymbol{\nu} \\ L_1 \end{bmatrix} A_{1,k} \\ &= \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ L_1 A_{1,k} \end{bmatrix}, \end{aligned} \tag{4.25}$$

for all $k = 2, \ldots, n$, It follows from (4.21a) and (4.24a) that

$$\begin{aligned} \tilde{L}_1 B_1 &= \begin{bmatrix} \boldsymbol{\mu} \\ \boldsymbol{\nu} \\ L_1 \end{bmatrix} B_1 \\ &= \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ L_1 B_1 \end{bmatrix}. \end{aligned} \tag{4.26}$$

Using equation (4.21) and (4.24b), we can obtain

$$\begin{aligned}
\tilde{L}_1 A_{1,1} \tilde{R}_1 &= \begin{bmatrix} \boldsymbol{\mu} \\ \boldsymbol{\nu} \\ L_1 \end{bmatrix} A_{1,1} \begin{bmatrix} \hat{\boldsymbol{\mu}} & \hat{\boldsymbol{\nu}} & R_1 \end{bmatrix} \\
&= \begin{bmatrix} \alpha_1 \boldsymbol{\mu} - \beta_1 \boldsymbol{\nu} \\ \beta_1 \boldsymbol{\mu} + \alpha_1 \boldsymbol{\nu} \\ L_1 A_{1,1} \end{bmatrix} \begin{bmatrix} \hat{\boldsymbol{\mu}} & \hat{\boldsymbol{\nu}} & R_1 \end{bmatrix} \\
&= \begin{bmatrix} \alpha_1 & -\beta_1 & \mathbf{0} \\ \beta_1 & \alpha_1 & \mathbf{0} \\ L_1 A_{1,1} \hat{\boldsymbol{\mu}} & L_1 A_{1,1} \hat{\boldsymbol{\nu}} & L_1 A_{1,1} R_1 \end{bmatrix}. \qquad (4.27)
\end{aligned}$$

By equation (4.21b), we have

$$C_1 \tilde{R}_1 = C_1 \begin{bmatrix} \hat{\boldsymbol{\mu}} & \hat{\boldsymbol{\nu}} & R_1 \end{bmatrix} = \begin{bmatrix} C_1 \hat{\boldsymbol{\mu}} & C_1 \hat{\boldsymbol{\nu}} & C_1 R_1 \end{bmatrix}, \qquad (4.28)$$

and

$$A_{k,1} \tilde{R}_1 = A_{k,1} \begin{bmatrix} \hat{\boldsymbol{\mu}} & \hat{\boldsymbol{\nu}} & R_1 \end{bmatrix} = \begin{bmatrix} A_{k,1} \hat{\boldsymbol{\mu}} & A_{k,1} \hat{\boldsymbol{\nu}} & A_{k,1} R_1 \end{bmatrix}. \qquad (4.29)$$

If let

$$\tilde{L} \triangleq \operatorname{diag}\{\tilde{L}_1, I_{r_2}, \ldots, I_{r_n}\}, \qquad (4.30a)$$

$$\tilde{R} \triangleq \tilde{L}^{-1} = \operatorname{diag}\{\tilde{R}_1, I_{r_2}, \ldots, I_{r_n}\}, \qquad (4.30b)$$

then we have

$$\tilde{L} A \tilde{R} = \begin{bmatrix} \alpha_1 & -\beta_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \beta_1 & \alpha_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ L_1 A_{1,1} \hat{\boldsymbol{\mu}} & L_1 A_{1,1} \hat{\boldsymbol{\nu}} & L_1 A_{1,1} R_1 & L_1 A_{1,2} & \cdots & L_1 A_{1,n} \\ A_{2,1} \hat{\boldsymbol{\mu}} & A_{2,1} \hat{\boldsymbol{\nu}} & A_{2,1} R_1 & A_{2,2} & \cdots & A_{2,n} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ A_{n,1} \hat{\boldsymbol{\mu}} & A_{n,1} \hat{\boldsymbol{\nu}} & A_{n,1} R_1 & A_{n,2} & \cdots & A_{n,n} \end{bmatrix}, \qquad (4.31a)$$

$$\tilde{L} B = \begin{bmatrix} \mathbf{0} & \mathbf{0} & (L_1 B_1)^{\mathrm{T}} & B_2^{\mathrm{T}} & \cdots & B_n^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}, \qquad (4.31b)$$

$$C \tilde{R} = \begin{bmatrix} C_1 \hat{\boldsymbol{\mu}} & C_1 \hat{\boldsymbol{\nu}} & C_1 R_1 & C_2 & \cdots & C_n \end{bmatrix}. \qquad (4.31c)$$

With attention

$$LAR = \begin{bmatrix} L_1A_{1,1}R_1 & L_1A_{1,2} & \cdots & L_1A_{1,n} \\ A_{2,1}R_1 & A_{2,2} & \cdots & A_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n,1}R_1 & A_{n,2} & \cdots & A_{n,n} \end{bmatrix}, \tag{4.32a}$$

$$LB = \begin{bmatrix} L_1B_1 \\ B_2 \\ \vdots \\ B_n \end{bmatrix}, \tag{4.32b}$$

$$CR = \begin{bmatrix} C_1R_1 & C_2 & \cdots & C_n \end{bmatrix}, \tag{4.32c}$$

and equation (4.30), then we have

$$\begin{aligned} CZ(I_r - AZ)^{-1}B =&(CZ)\tilde{R}\left(I_r - \tilde{L}AZ\tilde{R} \right)^{-1}\tilde{L}B \\ =&(C\tilde{R}Z)\left(I_r - \tilde{L}A\tilde{R}Z \right)^{-1}\tilde{L}B \\ =&(CR\hat{Z})\left(I_{\hat{r}} - LAR\hat{Z} \right)^{-1}LB, \end{aligned} \tag{4.33}$$

and $\hat{r}_1 = r_1 - 2$, $\hat{r}_k = r_k$ for all $k = 2, \dots, n$. That is to say, a new n-D Roesser model $(\hat{A}, \hat{B}, \hat{C}, D, \hat{\boldsymbol{r}})$ with lower order has been obtained. □

Remark 4.2. *Matrix $\tilde{L}_i$ defined in (4.20) can be obtained from a $r_i \times r_i$ identity matrix by replacing its k_0th and k_1th rows with $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ in (4.16), respectively.*

Remark 4.3. *Although Lemma 4.3 is based on a complex eigenvalue, it is also true for the real eigenvalues. If the eigenvalue is real, then $\tilde{M}_{i-}$ defined in (4.4) and $\boldsymbol{\omega}$ in (4.15) become real which gives $\boldsymbol{\omega} = \boldsymbol{\mu}$, $\boldsymbol{\nu} = \mathbf{0}$ and $k_1 = \emptyset$ (an empty element). Correspondingly, $\tilde{L}_i$ is obtained from a $r_i \times r_i$ identity matrix by just replacing its k_0th column with $\boldsymbol{\mu}$ and L_i can be obtained from $\tilde{L}_i$ just by deleting its k_0th row; $\tilde{R}_i = \tilde{L}_i^{-1}$ and R_i can be obtained from $\tilde{R}_i$ by just deleting its k_0th column, respectively. Finally, one can construct a new Roesser model $(\hat{A}, \hat{B}, \hat{C}, D) \triangleq (LAR, LB, CR, D)$ such that*

$$\hat{C}\hat{Z}(I_{\hat{r}} - \hat{A}\hat{Z})^{-1}\hat{B} = CZ(I_r - AZ)^{-1}B,$$

where $\hat{r}_i = r_i - 1$; $\hat{r}_k = r_k$ for all $k = 1, \dots, i-1, i+1, \dots, n$;

$$\hat{Z} = \mathrm{diag}\{z_1 I_{\hat{r}_1}, \dots, z_n I_{\hat{r}_n}\}; \tag{4.34}$$

Based on the above results, we can now give the following result.

Theorem 4.1. *If an n-D Roesser model is not eigenvalue trim or not eigenvalue co-trim, then it can be reduced.*

Proof. The result for eigenvalue co-trim is dual. Thus we just show that if an n-D Roesser model is not eigenvalue trim then it can be reduced.

If an n-D Roesser model $(A, B, C, D; \boldsymbol{r})$ is not eigenvalue trim, then there are $k \in \{1, \ldots, n\}$ and $t \in \{1, \ldots, l_k\}$ such that the matrix $\tilde{M}_{k-}$ in (4.4) is not full row rank. By Lemma 4.3, one can obtain a new lower-order Roesser model $(\hat{A}, \hat{B}, \hat{C}, D; \hat{\boldsymbol{r}})$ with $\hat{\boldsymbol{r}} = (\hat{r}_1, \ldots, \hat{r}_n)$ such that

$$\hat{C}\hat{Z}(I_{\hat{r}} - \hat{A}\hat{Z})^{-1}\hat{B} = CZ(I_r - AZ)^{-1}B, \tag{4.35}$$

where $\hat{Z} = \mathrm{diag}\{z_1 I_{\hat{r}_1}, \ldots, z_n I_{\hat{r}_n}\}$. □

Theorem 4.1 gives a sufficient condition for the reducibility of a given n-D Roesser model, that is, if it is not eigenvalue trim or not eigenvalue co-trim, then it can be reduced. Lemma 4.2 shows that an n-D Roesser model given by $(A, B, C, D; \boldsymbol{r})$ is eigenvalue trim if and only if with the partitions (4.3) we have that for $1 \le i \le n$, each matrix $\tilde{M}_{i-}$ in (4.4) is of full row rank for the distinct eigenvalues of $A_{i,i}$ with only real parts or with positive imaginary parts. Now, a procedure as described in Procedure 4.1, which can lower the order of an n-D Roesser model being not eigenvalue trim, is given, where $\lambda_{i,1}, \ldots, \lambda_{i,\hat{l}_i}$ denote the distinct eigenvalues of of $A_{i,i}$ in (3.12) with only real parts or with positive imaginary parts, where $i \in \{1, \ldots, n\}$, and $\hat{l}_i$ is the number of these distinct eigenvalues.

The details and effectiveness of the proposed algorithm are now illustrated by examples.

Example 4.1. *Consider the 2-D Roesser model $(A, B, C, D; \boldsymbol{r})$:*

$$A = \begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{bmatrix} = \left[\begin{array}{cccc|cc} -1 & 0 & 0 & -1 & -2 & 0 \\ 1 & -2 & -1 & 0 & -2 & 0 \\ -7 & 17 & 6 & -1 & 7 & -2 \\ 10 & 0 & 0 & 5 & 7 & -2 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right], \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \left[\begin{array}{cc} -1 & 0 \\ -1 & 0 \\ 3 & 0 \\ 3 & 0 \\ \hline 1 & 1 \\ 1 & 0 \end{array}\right], \tag{4.38}$$

$$C = [\, C_1 \;\; C_2 \,] = \left[\begin{array}{cccc|cc} 0 & 4 & 1 & 0 & 0 & 1 \\ -1 & 9 & 2 & 1 & 1 & 1 \end{array}\right], \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \boldsymbol{r} = (4,\ 2).$$

Procedure 4.1: Exact Order Reduction Based on Eigenvalue Trim

Input : A given n-D Roesser model $(A, B, C, D; \boldsymbol{r})$;
Output: A reduced-order n-D Roesser model $(\hat{A}, \hat{B}, \hat{C}, D; \hat{\boldsymbol{r}})$;

while $(A, B, C, D; \boldsymbol{r})$ *is not eigenvalue trim* **do**
  for *each* $i \in \{1, \ldots, n\}$ *and each* $t \in \{1, \ldots, \hat{l}_i\}$ **do**
    if $\lambda_{i,t}$ *is a real number* **then**
      Step 1A: Find a real vector $\boldsymbol{\omega}$ in the form of (4.15) such that the equation (4.17) holds. Find an index k_0 such that $\omega_{k_0} = 1$. Replacing the k_0th row of a $r_i \times r_i$ identity matrix by $\boldsymbol{\omega} \triangleq \boldsymbol{\mu}$ to construct the a matrix $\tilde{L}_i$ and obtain $\tilde{R}_i = \tilde{L}_i^{-1}$. Obtain R_i by deleting the k_0th row of $\tilde{R}_i$ and obtain L_i by deleting the k_0th column of $\tilde{L}_i$;
    else
      Step 1B: Find a complex vector $\boldsymbol{\omega}$ in the form of (4.15) which can be expressed by $\boldsymbol{\omega} \triangleq \boldsymbol{\mu} + j\boldsymbol{\nu}$ in the form of (4.16) such that the equation (4.17) holds. Find indices k_0 and k_1 such that $\omega_{k_0} = 1$ and $\nu_{k_1} \neq 0$. Replacing the k_0th and k_1th rows of a $r_i \times r_i$ identity matrix by $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$, respectively, to construct the a matrix in the form of (4.20) and obtain $\tilde{R}_i = \tilde{L}_i^{-1}$. Obtain R_i by deleting the k_0th and k_1th rows of $\tilde{R}_i$ and obtain L_i by deleting the k_0th and k_1th columns of $\tilde{L}_i$;
    end
  end
  Step 2: Construct

$$L \triangleq \mathrm{diag}(I_{r_1}, \ldots, L_i, \ldots, I_{r_n}), \quad R \triangleq \mathrm{diag}(I_{r_1}, \ldots, R_i, \ldots, I_{r_n}); \tag{4.36}$$

  Step 3: Obtain a new n-D Roesser state-space model $(\hat{A}, \hat{B}, \hat{C}, D; \hat{\boldsymbol{r}})$ with

$$\hat{A} \triangleq LAR, \quad \hat{B} \triangleq LB, \quad \hat{C} \triangleq CR \tag{4.37}$$

  and $\hat{\boldsymbol{r}} = (\hat{r}_1, \ldots \hat{r}_n)$;
  Renew $A = \hat{A}$, $B = \hat{B}$, $C = \hat{C}$, $\boldsymbol{r} = \hat{\boldsymbol{r}}$.
end
return $(\hat{A}, \hat{B}, \hat{C}, D; \hat{\boldsymbol{r}}) \triangleq (A, B, C, D; \boldsymbol{r})$,

Note for the complex eigenvalue $2 + j$ *of* $A_{1,1}$ *with positive imaginary part,*

$$\begin{aligned} \tilde{M}_{1-} &= \left[\begin{array}{ccc} A_{1,1} - (2+j)I_4 & A_{1,2} & B_1 \end{array}\right] \\ &= \left[\begin{array}{cccc|cc|cc} -3-j & 0 & 0 & -1 & -2 & 0 & -1 & 0 \\ 1 & -4-j & -1 & 0 & -2 & 0 & -1 & 0 \\ -7 & 17 & 4-j & -1 & 7 & -2 & 3 & 0 \\ 10 & 0 & 0 & 3-j & 7 & -2 & 3 & 0 \end{array}\right] \end{aligned} \tag{4.39}$$

has rank 3 *and is not of full row rank. Thus, the 2-D Roesser model of (4.38) is not eigenvalue trim, and then can be reduced by the proposed eigenvalue trim approach.*

For this model, $n = 2$, $\hat{l}_1 = 1$, $\hat{l}_2 = 2$, $\lambda_{1,1} = 2+j$, $\lambda_{2,1} = 0$ and $\lambda_{2,2} = 1$. For the $i = 1$, $t = 1$ and the complex eigenvalue $\lambda_{1,1} = 2 + j$, the specific steps are as follows.

Step 1B: Find a vector

$$\begin{aligned}\boldsymbol{\omega} &= \left[\begin{array}{cccc} 1 & -1 & -\frac{4}{17} - \frac{1}{17}j & \frac{4}{17} + \frac{1}{17}j \end{array}\right] \\ &= \left[\begin{array}{cccc} 1 & -1 & -\frac{4}{17} & \frac{4}{17} \end{array}\right] + j \left[\begin{array}{cccc} 0 & 0 & -\frac{1}{17} & +\frac{1}{17} \end{array}\right] \triangleq \boldsymbol{\mu} + j\boldsymbol{\nu} \end{aligned} \tag{4.40}$$

satisfying $\boldsymbol{\omega}\tilde{M}_{1-} = \mathbf{0}$. Since $\omega_1 = 1$ and $\nu_3 = -\frac{1}{17} \neq 0$, then one can set $k_0 = 1$ and $k_1 = 3$. Based on $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ in (4.40), we can construct

$$\begin{array}{cc} & \begin{array}{cccc} \downarrow \text{1st} & & \downarrow \text{3rd} & \end{array} \\ \tilde{L}_1 \triangleq & \left[\begin{array}{cccc} 1 & -1 & -\frac{4}{17} & \frac{4}{17} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{1}{17} & +\frac{1}{17} \\ 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \leftarrow \text{1st} \\ \\ \leftarrow \text{3rd'} \\ \\ \end{array} \end{array} \tag{4.41}$$

which is achieved from a 4×4 identity matrix with its 1st and 3rd rows replaced by $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$, respectively. Then, we can obtain

$$\tilde{R}_1 = \tilde{L}_1^{-1} = \left[\begin{array}{cccc} 1 & 1 & -4 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -17 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right]. \tag{4.42}$$

By deleting the 1st and 3rd rows of $\tilde{L}_1$, we obtain

$$L_1 = \left[\begin{array}{cccc} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]; \tag{4.43}$$

by deleting the 1st and 3rd columns of $\tilde{R}_1$, we obtain

$$R_1 = \left[\begin{array}{cc} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{array}\right]. \tag{4.44}$$

Step 2: Construct

$$L \triangleq \operatorname{diag}\{L_1, I_2\} = \left[\begin{array}{cccc|cc} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right], \tag{4.45a}$$

$$R \triangleq \operatorname{diag}\{R_1, I_2\} = \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]. \tag{4.45b}$$

Step 3: A new lower-order 2-D Roesser state-space model $(\hat{A}, \hat{B}, \hat{C}, D; \hat{\boldsymbol{r}})$ is obtained as

$$\begin{aligned}
\hat{A} =& LAR = \begin{bmatrix} \hat{A}_{1,1} & \hat{A}_{1,2} \\ \hat{A}_{2,1} & \hat{A}_{2,2} \end{bmatrix} = \left[\begin{array}{cc|cc} -1 & -1 & -2 & 0 \\ 10 & 5 & 7 & -2 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right], \\
\hat{B} =& LB = \begin{bmatrix} \hat{B}_1 \\ \hat{B}_2 \end{bmatrix} = \left[\begin{array}{cc} -1 & 0 \\ 3 & 0 \\ \hline 1 & 1 \\ 1 & 0 \end{array}\right], \\
\hat{C} =& CR = \begin{bmatrix} \hat{C}_1 & \hat{C}_2 \end{bmatrix} = \left[\begin{array}{cc|cc} 4 & 1 & 0 & 1 \\ 8 & 3 & 1 & 1 \end{array}\right], \\
\hat{D} =& D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \hat{\boldsymbol{r}} = (2, 2),
\end{aligned} \tag{4.46}$$

which is eigenvalue trim and eigenvalue co-trim.

It is seen that the order of the obtained Roesser model (4.46) is 4 ($\hat{\boldsymbol{r}} = (2, 2)$), which is lower than the order of 6 for the given Roesser model (4.38).

Remark 4.4. *It can be confirmed that the 2-D Roesser model (4.38) is trim and co-trim by Definition 1 or [23], and this Roesser model cannot be reduced by the trim approach given in [23] and elementary operation approach given in [32]. However, in Example 4.1 we have shown that this Roesser model is not eigenvalue trim, and thus can be reduced to a lower Roesser model as 4.46 by Procedure 4.1.*

The following example is given to show more details for real eigenvalue case.

Example 4.2. *Consider the 2-D Roesser model* $(A, B, C, D; \boldsymbol{r})$*:*

$$A = \begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{bmatrix} = \left[\begin{array}{ccccc|c} 2 & -1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 2 & -3 & 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 1 & 1 \\ \hline 2 & -2 & 0 & 1 & 0 & 2 \end{array}\right],$$

$$B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ \hline 0 & 0 \end{array}\right], \tag{4.47}$$

$$C = \begin{bmatrix} C_1 & C_2 \end{bmatrix} = \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 1 & 2 \end{array}\right],$$

$$D = \begin{bmatrix} 0 & 0 \end{bmatrix},$$

$$\boldsymbol{r} = (5,\ 1).$$

Note for the real eigenvalue 1 *of* $A_{1,1}$*, the matrix*

$$\begin{aligned} \tilde{M}_{1-} &= \begin{bmatrix} A_{1,1} - 1I_5 & A_{1,2} & B_1 \end{bmatrix} \\ &= \left[\begin{array}{ccccc|c|cc} 1 & -1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 2 & -3 & -1 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & 0 & -1 & 0 & 0 & 0 & 1 \\ 1 & -1 & 1 & 0 & 0 & 1 & 0 & 0 \end{array}\right] \end{aligned} \tag{4.48}$$

has rank 3 *and is not of full row rank. Thus, the 2-D Roesser model of (4.47) is not eigenvalue trim, and then can be reduced by the proposed eigenvalue trim approach.*

For this model, $n = 2$, $\hat{l}_1 = 2$, $\hat{l}_2 = 1$, $\lambda_{1,1} = 1$, $\lambda_{1,2} = 0$ and $\lambda_{2,1} = 2$. For $i = 1$, $t = 1$ and the real eigenvalue $\lambda_{1,1} = 1$, the specific steps are as follows.

Step 1A: Find a vector

$$\boldsymbol{\omega} = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \end{bmatrix} \tag{4.49}$$

such that $\boldsymbol{\omega}\tilde{M}_{1-} = \mathbf{0}$. Since $\omega_1 = 1$, one can set $k_0 = 1$. Then, letting $\boldsymbol{\mu} = \boldsymbol{\omega}$ and just replacing the 1st row of a 5×5 identity matrix by $\boldsymbol{\mu}$, we can construct

$$\tilde{L}_1 \triangleq \begin{array}{ll} \quad\downarrow \text{1st} & \\ \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} & \begin{array}{l} \leftarrow \text{1st} \\ \\ \\ \\ \\ \end{array} \end{array}. \tag{4.50}$$

Then, we obtain

$$\tilde{R}_1 = \tilde{L}_1^{-1} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{4.51}$$

We can further obtain

$$L_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \tag{4.52}$$

by deleting the 1st row of $\tilde{L}_1$, and

$$R_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{4.53}$$

by deleting the 1st column of $\tilde{R}_1$.

Step 2: Construct

$$L \triangleq \operatorname{diag}\{L_1, I_1\} = \left[\begin{array}{ccccc|c} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right], \tag{4.54a}$$

$$R \triangleq \operatorname{diag}\{R_1, I_1\} = \left[\begin{array}{ccccc} 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right]. \tag{4.54b}$$

Step 3: A new lower-order 2-D Roesser state-space model $(\hat{A}, \hat{B}, \hat{C}, D; \hat{\boldsymbol{r}})$ is obtained as

$$
\begin{aligned}
\hat{A} =& LAR = \begin{bmatrix} \hat{A}_{1,1} & \hat{A}_{1,2} \\ \hat{A}_{2,1} & \hat{A}_{2,2} \end{bmatrix} = \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ \hline 0 & 0 & 1 & 0 & 2 \end{array}\right], \\
\hat{B} =& LB = \begin{bmatrix} \hat{B}_1 \\ \hat{B}_2 \end{bmatrix} = \left[\begin{array}{cc} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ \hline 0 & 0 \end{array}\right], \\
\hat{C} =& CR = \begin{bmatrix} \hat{C}_1 & \hat{C}_2 \end{bmatrix} = \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 2 \end{array}\right], \\
\hat{D} =& D = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \\
\hat{\boldsymbol{r}} =& (4, 1).
\end{aligned}
\tag{4.55}
$$

Redefine $(A, B, C, D, \boldsymbol{r})$ as $(A, B, C, D, \boldsymbol{r}) \triangleq (\hat{A}, \hat{B}, \hat{C}, D, \hat{\boldsymbol{r}})$. One can verify that for the real eigenvalue 1 of $A_{1,1}$, the matrix

$$
\begin{aligned}
\tilde{M}_{1-} &= \begin{bmatrix} A_{1,1} - 1I_4 & A_{1,2} & B_1 \end{bmatrix} \\
&= \left[\begin{array}{cccc|c|cc} 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ -1 & -1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \end{array}\right]
\end{aligned}
\tag{4.56}
$$

is still not of full row rank, then Step 1A to Step 3 will be applied once again to generate a lower 2-D Roesser model $(\hat{A}, \hat{B}, \hat{C}, D, \hat{\boldsymbol{r}})$ as

$$
\begin{aligned}
&\hat{A} = \begin{bmatrix} \hat{A}_{1,1} & \hat{A}_{1,2} \\ \hat{A}_{2,1} & \hat{A}_{2,2} \end{bmatrix} = \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ \hline 0 & 1 & 0 & 2 \end{array}\right], \quad \hat{B} = \begin{bmatrix} \hat{B}_1 \\ \hat{B}_2 \end{bmatrix} = \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ \hline 0 & 0 \end{array}\right], \\
&\hat{C} = \begin{bmatrix} \hat{C}_1 & \hat{C}_2 \end{bmatrix} = \left[\begin{array}{ccc|c} 0 & 0 & 2 & 2 \end{array}\right], \quad \hat{D} = D = \begin{bmatrix} 0 & 0 \end{bmatrix}, \quad \hat{\boldsymbol{r}} = (3, 1),
\end{aligned}
\tag{4.57}
$$

which is now eigenvalue trim.

It is seen that the order of the obtained Roesser model (4.57) is 4 ($\hat{\boldsymbol{r}} = (3, 1)$), which is lower than the order of 6 for the given Roesser model (4.47).

4.2.1 Effective Reduction Procedure for Real Eigenvalues

In the previous subsection, reduction conditions and the corresponding reduction algorithm based on eigenvalue trim have been proposed for n-D Roesser models. From Remark 4.3 and Step 1A of Procedure 4.1, we see that the real eigenvalue case is treated in the same way for the complex eigenvalues case by just setting $k_1 = \emptyset$ when $\boldsymbol{\nu} = \mathbf{0}$, which is simple and easy to follow. However, it can be observed from Example 2 that if the difference between the size of $A_{i,i}$ and the rank of $\tilde{M}_{i-}$ in (4.4) is more than 1 for a certain real eigenvalue of $A_{i,i}$, then the corresponding order reduction has to be carried out by repeating Step 1A to Step 3 of Procedure 1 more than one time. In this subsection, we will show that a much more effective method for real eigenvalues can be established, which can directly achieve the order reduction without repeating these steps several times. That is, if the matrix $\tilde{M}_{i-}$ in (4.4) has row rank $\hat{r}_i$ for a real eigenvalue, we can directly reduce $r_i - \hat{r}_i$ orders of the given n-D Roesser model by executing the Procedure proposed below only once.

Before giving the new results, some preparations are needed. Let $\lambda_{i,t}$ be a real eigenvalue of the matrix $A_{i,i}$, $i \in \{1, \ldots, n\}$. The first nonzero entry of a certain column or row in a given matrix is called the leading entry of the column or row [68]. Denote by $\rho(\cdot)$ the operator transforming the (real) matrices $\tilde{M}_{i-}$ in (4.4), $i = 1, \ldots, n$, to the reduced column echelon form by elementary column transformation, i.e.,

$$\rho(\tilde{M}_{i-}) \triangleq \tilde{M}_{i-} P_i = \left[\begin{array}{cccccc|c} 1 & 0 & 0 & \cdots & 0 & \mathbf{0} \\ * & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ 0 & 1 & 0 & \cdots & 0 & \mathbf{0} \\ * & * & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ 0 & 0 & 1 & \mathbf{0} & 0 & \mathbf{0} \\ * & * & * & \cdots & \mathbf{0} & \mathbf{0} \\ 0 & 0 & 0 & \cdots & 1 & \mathbf{0} \\ * & * & * & \cdots & * & \mathbf{0} \end{array} \right] \triangleq \left[\begin{array}{c|c} R_i & \mathbf{0} \end{array} \right], \tag{4.58}$$

where $*$ denotes a (real) column vector, $\mathbf{0}$ denotes a zero vector or matrix of suitable size, and P_i denotes the elementary column transformation [68].

The above reduced column echelon form has the following properties [68, 69]:

(a) Nonzero columns precede zero columns;

(b) The leading entry of any nonzero column is 1;

(c) The leading entries occur in a stairstep pattern, left to right; that is, the leading entry in a nonzero column is below the leading entries in preceding columns and above the leading entries of succeeding columns;

(d) In each row which contains the leading entry 1 of some column, the entries preceding that leading entry are zero.

Let $\hat{\rho}(\tilde{M}_{i-})$ be an operator consisting of the operations: first conduct $\rho(\tilde{M}_{i-})$, then set the non-leading entries of nonzero columns in $\rho(\tilde{M}_{i-})$, i.e., those denoted by $*$ in (4.58), to 0. That is, for the matrices $\tilde{M}_{i-}$ in (4.4), $i = 1, \ldots, n$, we have

$$\hat{\rho}(\tilde{M}_{i-}) \triangleq \left[\begin{array}{cccccc|c} 1 & 0 & 0 & \cdots & 0 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ 0 & 1 & 0 & \cdots & 0 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ 0 & 0 & 1 & \mathbf{0} & 0 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ 0 & 0 & 0 & \cdots & 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \end{array}\right] \triangleq \left[\begin{array}{cc} L_i & \mathbf{0} \end{array}\right]. \tag{4.59}$$

Let $\hat{r}_i = \mathrm{rank}(\tilde{M}_{i-})$ with $\tilde{M}_{i-}$ defined in (4.4), and note that

$$\mathrm{rank}(\tilde{M}_{i-}) = \mathrm{rank}(\rho(\tilde{M}_{i-})) = \mathrm{rank}(R_i) = \mathrm{rank}(L_i),$$

then $R_i,\ L_i \in \mathbf{R}^{r_i \times \hat{r}_i}$. Observing the definitions of $\hat{\rho}(\tilde{M}_{i-})$ and L_i in (4.59) and $\rho(\tilde{M}_{i-})$ in (4.58), it can be verified that

$$L_i^{\mathrm{T}} \rho(\tilde{M}_{i-}) = L_i^{\mathrm{T}} \tilde{M}_{i-} P_i = \left[\begin{array}{cc} I_{\hat{r}_i} & \mathbf{0} \end{array}\right]. \tag{4.60}$$

Now, if we define the transformation matrices by

$$L \triangleq \ \mathrm{diag}\{I_{r_1}, \ldots, L_i, \ldots, I_{r_n}\}, \quad R \triangleq \ \mathrm{diag}\{I_{r_1}, \ldots, R_i, \ldots, I_{r_n}\}, \tag{4.61}$$

then we have the following lemma.

Lemma 4.4. *For a given n-D Roesser model* $(A, B, C, D; \boldsymbol{r})$*, construct a new Roesser model* $(\hat{A}, \hat{B}, \hat{C}, D; \hat{\boldsymbol{r}})$ *with*

$$\hat{A} \triangleq LAR, \quad \hat{B} \triangleq LB, \quad \hat{C} \triangleq CR \tag{4.62}$$

where L and R defined in (4.61) and $\hat{\mathbf{r}} = (\hat{r}_1, \dots \hat{r}_n)$ with $\hat{r}_k = r_k$ for all $k = 1, \dots, i-1, i+1, \dots, n$. Then, we have

$$\hat{C}\hat{Z}(I_{\hat{r}} - \hat{A}\hat{Z})^{-1}\hat{B} = CZ(I_r - AZ)^{-1}B,$$

where

$$\hat{Z} = \mathrm{diag}\{z_1 I_{\hat{r}_1}, \dots, z_n I_{\hat{r}_n}\}. \tag{4.63}$$

Proof. Let

$$\Lambda = \mathrm{diag}\{0I_{r_1}, \dots, \lambda_{i,t} I_{r_i}, \dots, 0I_{r_n}\} \tag{4.64}$$

and

$$\hat{\Lambda} = L\Lambda R = \mathrm{diag}\{0I_{r_1}, \dots, \lambda_{i,t} I_{\hat{r}_i}, \dots, 0I_{r_n}\}.$$

By (4.60), we can have

$$R_i L_i^{\mathrm{T}} \tilde{M}_{i-} P_i = R_i (L_i^{\mathrm{T}} \tilde{M}_{i-} P_i) = R_i \begin{bmatrix} I_{\hat{r}_i} & \mathbf{0} \end{bmatrix} = \begin{bmatrix} R_i & \mathbf{0} \end{bmatrix} = \tilde{M}_{i-} P_i. \tag{4.65}$$

Multiplying (4.65) by P_i^{-1} from the right side gives that

$$R_i L_i^{\mathrm{T}} \tilde{M}_{i-} = \tilde{M}_{i-}, \tag{4.66}$$

or

$$R_i L_i^{\mathrm{T}} \begin{bmatrix} (A_{i-} - \lambda_{i,t} I_{i-}) & B_i \end{bmatrix} = \begin{bmatrix} (A_{i-} - \lambda_{i,t} I_{i-}) & B_i \end{bmatrix}. \tag{4.67}$$

Therefore,

$$RL(A - \Lambda) = (A - \Lambda), \quad RLB = B. \tag{4.68}$$

Note that

$$R\hat{Z}^{-1} = Z^{-1}R, \quad R\hat{\Lambda} = \Lambda R,$$

hence

$$R\left(\hat{Z}^{-1} - \hat{\Lambda} - L(A - \Lambda)R\right) = \left(Z^{-1} - \Lambda - RL(A - \Lambda)\right)R \tag{4.69}$$

or

$$\left(Z^{-1} - \Lambda - RL(A - \Lambda)\right)^{-1} R = R\left(\hat{Z}^{-1} - \hat{\Lambda} - L(A - \Lambda)R\right)^{-1} \tag{4.70}$$

Therefore, by equation (4.68) and (4.70) we have

$$\begin{aligned}
& CZ(I_r - AZ)^{-1}B \\
=& C\left(Z^{-1} - A\right)^{-1} B \\
=& C\left(Z^{-1} - \Lambda - (A - \Lambda)\right)^{-1} B \\
=& C\left(Z^{-1} - \Lambda - RL\left(A - \Lambda\right)\right)^{-1} RLB \\
=& CR\left(\hat{Z}^{-1} - L\Lambda R - L\left(A - \Lambda\right)R\right)^{-1} LB \\
=& CR\left(\hat{Z}^{-1} - LAR\right)^{-1} LB = \hat{C}(\hat{Z}^{-1} - \hat{A})^{-1}\hat{B} = \hat{C}\hat{Z}(I_{\hat{r}} - \hat{A}\hat{Z})^{-1}\hat{B}.
\end{aligned} \tag{4.71}$$

□

Based on the above results, Step 1A of Procedure 4.1 can be replaced by Step 1A$'$ given below.

Step 1A$'$: Computing $\rho(\tilde{M}_{i-})$ in (4.58) and $\hat{\rho}(\tilde{M}_{i-})$ in (4.59) to achieve L_i and R_i;

Remark 4.5. *In Step 1A of Procedure 4.1, to construct L_i and R_i, we have to find a vector $\boldsymbol{\omega}$ such that $\boldsymbol{\omega}\tilde{M}_{i-} = 0$, which means that $\boldsymbol{\omega}^{\mathrm{T}}$ is a vector of the null space of $\tilde{M}_{i-}^{\mathrm{T}}$. It is well known that a null space basis of $\tilde{M}_{i-}^{\mathrm{T}}$ can be obtained by transforming $\tilde{M}_{i-}^{\mathrm{T}}$ to its echelon form [70, 71]. Moreover, as discussed previously, this process may be repeated several times when conducting practical order reduction, that is, we may need to find a series of L_i and R_i. In contrast, in Step 1A$'$, we only need to construct L_i and R_i once directly from the reduced echelon form of $\tilde{M}_{i-}$. Therefore, Step 1A$'$ is obviously much more effective and efficient than Step 1A.*

The effectiveness of Step 1A$'$ is illustrated by the following example.

Example 4.3. *Consider again the 2-D Roesser model $(A, B, C, D; \boldsymbol{r})$ in (4.47) in Example 4.2*

For this model, $n = 2$, $\hat{l}_1 = 2$, $\hat{l}_2 = 1$, $\lambda_{1,1} = 1$, $\lambda_{1,2} = 0$ and $\lambda_{2,1} = 2$. For $i = 1$, $t = 1$ and the real eigenvalue $\lambda_{1,1} = 1$, the specific steps are as follows.

Step 1: Computing

$$\rho(\tilde{M}_{1-}) = \left[\begin{array}{ccc|ccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \triangleq \left[\begin{array}{cc} R_1 & \mathbf{0}_{5\times 5} \end{array}\right], \tag{4.72}$$

to achiever R_1, and computing

$$\hat{\rho}(\tilde{M}_{1-}) = \left[\begin{array}{ccc|ccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \triangleq \left[\begin{array}{cc} L_1 & \mathbf{0}_{5\times 5} \end{array}\right], \tag{4.73}$$

to achieve L_1.

Step 2: Construct

$$L = \mathrm{diag}\{L_1, I_1\}^{\mathrm{T}} = \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right], \quad R = \mathrm{diag}\{R_1, I_1\} = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]. \tag{4.74}$$

Step 3: A new lower-order 2-D Roesser state-space model $(\hat{A}, \hat{B}, \hat{C}, D; \hat{\boldsymbol{r}})$ is obtained as

$$\begin{aligned} \hat{A} =& LAR = \left[\begin{array}{cc} \hat{A}_{1,1} & \hat{A}_{1,2} \\ \hat{A}_{2,1} & \hat{A}_{2,2} \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 2 \end{array}\right], \\ \hat{B} =& LB = \left[\begin{array}{c} \hat{B}_1 \\ \hat{B}_2 \end{array}\right] = \left[\begin{array}{cc} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ \hline 0 & 0 \end{array}\right], \\ \hat{C} =& CR = \left[\begin{array}{cc} \hat{C}_1 & \hat{C}_2 \end{array}\right] = \left[\begin{array}{ccc|c} 2 & 0 & 0 & 2 \end{array}\right], \\ D =& \left[\begin{array}{cc} 0 & 0 \end{array}\right], \quad \hat{\boldsymbol{r}} = (3, 1), \end{aligned} \tag{4.75}$$

which is in eigenvalue trim and eigenvalue co-trim form.

Remark 4.6. *It is seen that by Step 1A$'$ one can directly!! reduce the given 2-D Roesser model by 2 orders and obtain the new 2-D Roesser model (4.75) with order 4 ($\hat{\boldsymbol{r}} = (3,1)$).*

4.3 Equivalent Transformation between Different Blocks

In the previous section, the exact order reduction is studied only in a certain block matrix, i.e., in $\tilde{M}_{i-}$ defined in (4.4). Note that even if the obtained n-D Roesser model is eigenvalue trim and eigenvalue co-trim, this Roesser model may not be minimal or absolutely minimal in general. Therefore, to achieve further exact order reduction of n-D systems, it is necessary to consider the relationship between different blocks that correspond to different variables.

In this section, a transformation will be introduced by swapping certain columns and rows and interchanging certain entries that belong to different blocks. This transformation can make further reduction possible, since it may change the property of the original Roesser model being eigenvalue trim and eigenvalue co-trim.

Let $a_{\xi,\eta}$ be the (ξ,η)th entry of A, and $\Pi_{\xi,\eta}$ denotes the permutation matrix that interchanges the ξth and ηth rows of A if used in the form $\Pi_{\xi,\eta}A$ (or columns if used in the form $A\Pi_{\xi,\eta}$, respectively). $\Theta_{\xi,\eta}(A)$ denotes the operation of interchanging the entries $a_{\xi,\xi}$ and $a_{\eta,\eta}$ of A. Then, we have the following results.

Theorem 4.2. *For a given Roesser model $(A,B,C,D;\boldsymbol{r})$ with $\boldsymbol{r} = (r_1,\ldots,r_n)$, if there is a pair of indices (ξ,η) with $1 \le \xi,\eta \le r$ such that*

$$a_{\xi,j} = 0 \quad \textit{for all } j \neq \eta, \xi; \tag{4.76a}$$

$$\textit{and} \quad a_{k,\eta} = 0 \quad \textit{for all } k \neq \xi; \tag{4.76b}$$

$$\textit{and} \quad b_{\xi,j} = 0 \quad \textit{for all } j; \tag{4.76c}$$

$$\textit{and} \quad c_{k,\eta} = 0 \quad \textit{for all } k. \tag{4.76d}$$

Then, we have

$$CZ(I-AZ)^{-1}B = C\Pi_{\xi,\eta}Z(I-\Pi_{\xi,\eta}\Theta_{\xi,\eta}(A)\Pi_{\xi,\eta})^{-1}\Pi_{\xi,\eta}B, \tag{4.77}$$

with $Z = \operatorname{diag}\{z_1 I_{r_1},\ldots,z_n I_{r_n}\}$.

Proof. Note that a Roesser model (A, B, C, D) satisfying (4.76) is in the form as

$$A = \begin{bmatrix} X_{1,1} & \mathbf{0} & X_{1,3} & X_{1,4} & X_{1,5} \\ X_{2,1} & 0 & X_{2,3} & a_{\eta,\xi} & X_{2,5} \\ X_{3,1} & \mathbf{0} & X_{3,3} & X_{3,4} & X_{3,5} \\ \mathbf{0} & a_{\xi,\eta} & \mathbf{0} & a_{\xi,\xi} & \mathbf{0} \\ X_{5,1} & \mathbf{0} & X_{5,3} & X_{5,4} & X_{5,5} \end{bmatrix},$$

$$B = \begin{bmatrix} X_{b1} \\ X_{b2} \\ X_{b3} \\ \mathbf{0} \\ X_{b5} \end{bmatrix}, \tag{4.78}$$

$$C = \begin{bmatrix} X_{c1} & \mathbf{0} & X_{c3} & X_{c4} & X_{c5} \end{bmatrix}.$$

where $X_{k,j}$, X_{bj}, and X_{cj} denote the block matrices with appropriate dimensions in A, B and C, respectively, $k = 1, \dots, 5$, $j = 1, \dots, 5$. The expected Roesser model $(\hat{A}, \hat{B}, \hat{C}, D; \boldsymbol{r}) = (\Pi_{\xi,\eta}\Theta_{\xi,\eta}(A)\Pi_{\xi,\eta}, \Pi_{\xi,\eta}B, C\Pi_{\xi,\eta}, D; \boldsymbol{r})$ is in the form as

$$\hat{A} = \begin{bmatrix} X_{1,1} & X_{1,4} & X_{1,3} & \mathbf{0} & X_{1,5} \\ \mathbf{0} & 0 & \mathbf{0} & a_{\xi,\eta} & \mathbf{0} \\ X_{3,1} & X_{3,4} & X_{3,3} & \mathbf{0} & X_{3,5} \\ X_{2,1} & a_{\eta,\xi} & X_{23} & a_{\xi,\xi} & X_{2,5} \\ X_{5,1} & X_{5,4} & X_{5,3} & \mathbf{0} & X_{5,5} \end{bmatrix},$$

$$\hat{B} = \begin{bmatrix} X_{b1} \\ \mathbf{0} \\ X_{b3} \\ X_{b2} \\ X_{b5} \end{bmatrix}, \tag{4.79}$$

$$\hat{C} = \begin{bmatrix} X_{c1} & X_{c4} & X_{c3} & \mathbf{0} & X_{c5} \end{bmatrix}.$$

Let

$$Z = \mathrm{diag}\{z_1 I_{r_1}, \dots, z_n I_{r_n}\} \tag{4.80}$$

and $z_1, \dots, z_n$ denote the unit delay (backward-shift) operators. Compatible with A, Z can be partitioned as

$$Z = \mathrm{diag}\{Z_1, Z_2, Z_3, Z_4, Z_5\} \tag{4.81}$$

where the sizes of $Z_1, \dots, Z_5$ are respectively, $l_1, 1, l_3, 1, l_5$. The system equations of the

n-D Roesser model (4.78) are in the following form

$$\boldsymbol{w}_1 = X_{1,1}Z_1\boldsymbol{w}_1 + X_{1,3}Z_3\boldsymbol{w}_3 + X_{1,4}Z_4w_4 + X_{1,5}Z_5\boldsymbol{w}_5 + X_{b1}\boldsymbol{u}_1, \tag{4.82a}$$

$$w_2 = X_{2,1}Z_1\boldsymbol{w}_1 + X_{2,3}Z_3\boldsymbol{w}_3 + a_{\eta,\xi}Z_4w_4 + X_{2,5}Z_5\boldsymbol{w}_5 + X_{b2}\boldsymbol{u}_2, \tag{4.82b}$$

$$\boldsymbol{w}_3 = X_{3,1}Z_1\boldsymbol{w}_1 + X_{3,3}Z_3\boldsymbol{w}_3 + X_{3,4}Z_4w_4 + X_{3,5}Z_5\boldsymbol{w}_5 + X_{b3}\boldsymbol{u}_3, \tag{4.82c}$$

$$w_4 = a_{\xi,\eta}Z_2w_2 + a_{\xi,\xi}Z_4w_4, \tag{4.82d}$$

$$\boldsymbol{w}_5 = X_{5,1}Z_1\boldsymbol{w}_1 + X_{5,3}Z_3\boldsymbol{w}_3 + X_{5,4}Z_4w_4 + X_{5,5}Z_5\boldsymbol{w}_5 + X_{b5}\boldsymbol{u}_5, \tag{4.82e}$$

$$\boldsymbol{y} = C_{c1}Z_1\boldsymbol{w}_1 + C_{c3}Z_3\boldsymbol{w}_3 + C_{c4}Z_4w_4 + C_{c5}Z_5\boldsymbol{w}_5 + D\boldsymbol{u}. \tag{4.82f}$$

with

$$\boldsymbol{u} \triangleq \begin{bmatrix} \boldsymbol{u}_1 \\ u_2 \\ \boldsymbol{u}_3 \\ u_4 \\ \boldsymbol{u}_5 \end{bmatrix}.$$

Equation (4.82d) is equivalent to the following equation

$$w_4 = \frac{a_{\xi,\eta}Z_2w_2}{1 - a_{\xi,\xi}Z_4}. \tag{4.83}$$

Now, Let

$$\tilde{w}_2 = \frac{a_{\xi,\eta}Z_4w_2}{1 - a_{\xi,\xi}Z_4}, \tag{4.84}$$

which can be expressed as

$$\tilde{w}_2 = a_{\xi,\eta}Z_4w_2 + a_{\xi,\xi}Z_4\tilde{w}_2. \tag{4.85}$$

Then, we have

$$Z_2\tilde{w}_2 = Z_2Z_4\frac{a_{\xi,\eta}w_2}{1 - a_{\xi,\xi}Z_4} = Z_4Z_2\frac{a_{\xi,\eta}w_2}{1 - a_{\xi,\xi}Z_4} = Z_4w_4. \tag{4.86}$$

In attention to equations (4.85) and (4.86), the system equations in (4.82) are equivalent to

$$\boldsymbol{w}_1 = X_{1,1}Z_1\boldsymbol{w}_1 + X_{1,3}Z_3\boldsymbol{w}_3 + X_{1,4}Z_2\tilde{w}_2 + X_{1,5}Z_5\boldsymbol{w}_5 + X_{b1}\boldsymbol{u}_1, \tag{4.87a}$$

$$w_2 = X_{2,1}Z_1\boldsymbol{w}_1 + X_{2,3}Z_3\boldsymbol{w}_3 + a_{\eta,\xi}Z_2\tilde{w}_2 + X_{2,5}Z_5\boldsymbol{w}_5 + X_{b2}\boldsymbol{u}_2, \tag{4.87b}$$

$$\boldsymbol{w}_3 = X_{3,1}Z_1\boldsymbol{w}_1 + X_{3,3}Z_3\boldsymbol{w}_3 + X_{3,4}Z_2\tilde{w}_2 + X_{3,5}Z_5\boldsymbol{w}_5 + X_{b3}\boldsymbol{u}_3, \tag{4.87c}$$

$$\tilde{w}_2 = a_{\xi,\eta}Z_4 w_2 + a_{\xi,\xi}Z_4\tilde{w}_2, \tag{4.87d}$$

$$\boldsymbol{w}_5 = X_{5,1}Z_1\boldsymbol{w}_1 + X_{5,3}Z_3\boldsymbol{w}_3 + X_{5,4}Z_2\tilde{w}_2 + X_{5,5}Z_5\boldsymbol{w}_5 + X_{b5}\boldsymbol{u}_5, \tag{4.87e}$$

$$\boldsymbol{y} = C_{c1}Z_1\boldsymbol{w}_1 + C_{c3}Z_3\boldsymbol{w}_3 + C_{c4}Z_2\tilde{w}_2 + C_{c5}Z_5\boldsymbol{w}_5 + D\boldsymbol{u}. \tag{4.87f}$$

From equation (4.87d), we can obtain that

$$\tilde{w}_2 = a_{\xi,\eta}Z_4\frac{w_2}{1 - a_{\xi,\xi}Z_4}. \tag{4.88}$$

If we let

$$\begin{aligned}\hat{w}_4 &= \frac{w_2}{1 - a_{\xi,\xi}Z_4},\\ \hat{w}_2 &= \tilde{w}_2,\end{aligned} \tag{4.89}$$

then, we have

$$\hat{w}_2 = a_{\xi,\eta}Z_4\hat{w}_4, \tag{4.90a}$$

$$\hat{w}_4 = w_2 + a_{\xi,\xi}Z_4\hat{w}_4. \tag{4.90b}$$

By equations (4.90b) and (4.87b), we can get

$$\begin{aligned}\hat{w}_4 &= w_2 + a_{\xi,\xi}Z_4\hat{w}_4\\ &= X_{2,1}Z_1\boldsymbol{w}_1 + X_{2,3}Z_3\boldsymbol{w}_3 + a_{\eta,\xi}Z_2\tilde{w}_2 + X_{2,5}Z_5\boldsymbol{w}_5 + X_{b2}\boldsymbol{u}_2 + a_{\xi,\xi}Z_4\hat{w}_4\\ &= X_{2,1}Z_1\boldsymbol{w}_1 + a_{\eta,\xi}Z_2\hat{w}_2 + X_{2,3}Z_3\boldsymbol{w}_3 + a_{\xi,\xi}Z_4\hat{w}_4 + X_{2,5}Z_5\boldsymbol{w}_5 + X_{b2}\boldsymbol{u}_2.\end{aligned} \tag{4.91}$$

In view of equations (4.90) and (4.91), the system equations in (4.87) are equivalent to the following equations

$$\boldsymbol{w}_1 = X_{1,1}Z_1\boldsymbol{w}_1 + X_{1,4}Z_2\hat{w}_2 + X_{1,3}Z_3\boldsymbol{w}_3 + X_{1,5}Z_5\boldsymbol{w}_5 + X_{b1}\boldsymbol{u}_1, \tag{4.92a}$$

$$\hat{w}_2 = a_{\xi,\eta}Z_4\hat{w}_4, \tag{4.92b}$$

$$\boldsymbol{w}_3 = X_{3,1}Z_1\boldsymbol{w}_1 + X_{3,4}Z_2\hat{w}_2 + X_{3,3}Z_3\boldsymbol{w}_3 + X_{3,5}Z_5\boldsymbol{w}_5 + X_{b3}\boldsymbol{u}_3, \tag{4.92c}$$

$$\hat{w}_4 = X_{2,1}Z_1\boldsymbol{w}_1 + a_{\eta,\xi}Z_2\hat{w}_2 + X_{2,3}Z_3\boldsymbol{w}_3 + a_{\xi,\xi}Z_4\hat{w}_4 + X_{2,5}Z_5\boldsymbol{w}_5 + X_{b2}\boldsymbol{u}_2, \tag{4.92d}$$

$$\boldsymbol{w}_5 = X_{5,1}Z_1\boldsymbol{w}_1 + X_{5,4}Z_2\hat{w}_2 + X_{5,3}Z_3\boldsymbol{w}_3 + X_{5,5}Z_5\boldsymbol{w}_5 + X_{b5}\boldsymbol{u}_5, \tag{4.92e}$$

$$\boldsymbol{y} = C_{c1}Z_1\boldsymbol{w}_1 + C_{c4}Z_2\hat{w}_2 + C_{c3}Z_3\boldsymbol{w}_3 + C_{c5}Z_5\boldsymbol{w}_5 + D\boldsymbol{u}, \tag{4.92f}$$

which correspond to the expected Roesser model $(\hat{A}, \hat{B}, \hat{C}, D; \boldsymbol{r})$ in (4.79). Thus, the Roesser models (4.78) and (4.79) represent the same system. Therefore,

$$CZ(I - AZ)^{-1}B = C\Pi_{\xi,\eta}Z(I - \Pi_{\xi,\eta}\Theta_{\xi,\eta}(A)\Pi_{\xi,\eta})^{-1}\Pi_{\xi,\eta}B.$$

□

Remark 4.7. *The key point here is that the ξth and ηth rows of Z have different variables. Otherwise, i.e., for the case that the ξth and ηth rows of Z have the same variable, $\Pi_{\xi,\eta}$ and $\Theta_{\xi,\eta}(A)$ will not affect the property on reducibility.*

From the process of proving that the given n-D Roesser model $(A, B, C, D; \boldsymbol{r})$ in the form of (4.78) is equivalent to the n-D Roesser model $(\Pi_{\xi,\eta}\Theta_{\xi,\eta}(A)\Pi_{\xi,\eta}, \Pi_{\xi,\eta}B, C\Pi_{\xi,\eta}, D)$ in (4.79), it is easy to see that the reverse is also true. Then, we have the following theorem.

Theorem 4.3. *For a given Roesser model $(A, B, C, D; \boldsymbol{r})$ with $\boldsymbol{r} = (r_1, \ldots, r_n)$, if there is a pair of indices (ξ, η) with $1 \leq \xi, \eta \leq r$ such that*

$$\begin{aligned} & a_{\xi,j} = 0 \quad \textit{for all } j \neq \eta; \\ \textit{and} \quad & a_{k,\eta} = 0 \quad \textit{for all } k \neq \eta, \xi; \\ \textit{and} \quad & b_{\xi,j} = 0 \quad \textit{for all } j; \\ \textit{and} \quad & c_{k,\eta} = 0 \quad \textit{for all } k. \end{aligned} \tag{4.93}$$

Then, we have

$$CZ(I - AZ)^{-1}B = C\Pi_{\xi,\eta}Z(I - \Pi_{\xi,\eta}\Theta_{\xi,\eta}(A)\Pi_{\xi,\eta})^{-1}\Pi_{\xi,\eta}B, \tag{4.94}$$

with $Z = \mathrm{diag}\{z_1 I_{r_1}, \ldots, z_n I_{r_n}\}$.

Proof. The proof is dual to the one for Theorem 4.2 as mentioned above, and thus is omitted. ☐

Example 4.4. *Consider the 2-D Roesser model* $(\hat{A}, \hat{B}, \hat{C}, D; \boldsymbol{r})$ *in (4.75) obtained from (4.47) in Example 4.2 which is both eigenvalue trim and eigenvalue co-trim, and redefine* $(A, B, C, D; \boldsymbol{r})$ *as* $(A, B, C, D; \boldsymbol{r}) = (\hat{A}, \hat{B}, \hat{C}, D; \boldsymbol{r})$. *There is a pair of indices* $(\xi, \eta) = (3, 4)$ *which meets the conditions in (4.76) of Theorem 4.2.*

$\Theta_{4,2}(A)$ gives that

$$\Theta_{3,4}(A) = \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ \hline 0 & 0 & 1 & 0 \end{array}\right]. \tag{4.95}$$

By Lemma 4.2, the Roesser model of $(A, B, C, D; \boldsymbol{r})$ is equivalent to the following Roesser state-space model:

$$\begin{aligned} \hat{A} =& \Pi_{3,4}\Theta_{3,4}(A)\Pi_{3,4} = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 2 \end{array}\right], \quad \hat{B} = \Pi_{3,4}B = \left[\begin{array}{cc} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ \hline 0 & 1 \end{array}\right], \\ \hat{C} =& C\Pi_{3,4} = \left[\begin{array}{ccc|c} 2 & 0 & 2 & 0 \end{array}\right], \quad D = \left[\begin{array}{cc} 0 & 0 \end{array}\right], \end{aligned} \tag{4.96}$$

which is not eigenvalue co-trim.

It is seen that after applying the proposed transformation, the Roesser model in (4.75), which is eigenvalue trim and eigenvalue co-trim, is transformed to the new Roesser model $(A, B, C, D; \boldsymbol{r})$ in (4.96) that is no longer eigenvalue co-trim.

Example 4.5. *Consider the* 2*-D Roesser model* $(A, B, C, D;\, \boldsymbol{r}) \triangleq (\hat{A}, \hat{B}, \hat{C}, D; \boldsymbol{r})$ *of (4.96) obtained in Example 4.4, which is not eigenvalue co-trim.*

Since the Roesser model $(A, B, C, D; \boldsymbol{r})$ is not eigenvalue co-trim, then the Roesser model $(A^*, B^*, C^*, D^*; \boldsymbol{r}) = (A^{\mathrm{T}}, C^{\mathrm{T}}, B^{\mathrm{T}}, D^{\mathrm{T}}, \boldsymbol{r})$ with

$$A^* = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 2 \end{array}\right], \quad B^* = \left[\begin{array}{c} 2 \\ 0 \\ 2 \\ \hline 0 \end{array}\right], \quad C^* = \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right], \quad D^* = \left[\begin{array}{c} 0 \\ 0 \end{array}\right], \tag{4.97}$$

is not eigenvalue trim due to the duality of eigenvalue trim and eigenvalue co-trim.

By Procedure 1, one can reduce the 2-D Roesser model $(A^*, B^*, C^*, D^*; \boldsymbol{r})$ to a new 2-D Roesser model:

$$\begin{aligned} \hat{A}^* &= \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 1 & 0 & 0 \\ \hline 1 & 0 & 2 \end{array}\right], \quad \hat{B}^* = \left[\begin{array}{c} 2 \\ 0 \\ \hline 0 \end{array}\right], \\ \hat{C}^* &= \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right], \quad D^* = \left[\begin{array}{c} 0 \\ 0 \end{array}\right], \quad \hat{\boldsymbol{r}} = (3, 1), \end{aligned} \tag{4.98}$$

which is eigenvalue trim. Then, one can obtain an eigenvalue co-trim Roesser model as

$$\begin{aligned} \hat{A} =& \hat{A}^{*\mathrm{T}} = \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 2 \end{array}\right], \quad \hat{B} = \hat{C}^{*\mathrm{T}} = \left[\begin{array}{cc} 0 & 0 \\ 1 & 0 \\ \hline 0 & 1 \end{array}\right], \\ \hat{C} =& \hat{B}^{*\mathrm{T}} = \left[\begin{array}{cc|c} 2 & 0 & 0 \end{array}\right], \quad D = \left[0\,0\right], \quad \hat{\boldsymbol{r}} = (2, 1), \end{aligned} \tag{4.99}$$

for the given Roesser model of (4.96).

Remark 4.8. *It should be noted that the 2-D Roesser model of (4.75) cannot be transformed to another Roesser state-space model by the transformation introduced in [23], and thus cannot be reduced by the reduction techniques in [4, 26, 29–32, 34].*

4.4 Further Comparisons and Application Examples

To see some more details on the effectiveness of our new method, in this section we first give further comparisons to the representative exact order reduction approaches including the trim approach [23, 24], the elementary operation approach [32], and the n-D Jordan transformation approach [29]. Then, two application examples will also be presented.

4.4.1 Further Comparisons to Existing Results

As clarified in [32], the reduction conditions of the trim reduction approach and the elementary operation reduction approach in one block matrix require that the given Roesser model is not in trim form or not in co-trim form. As discussed above, eigenvalue trim and eigenvalue co-trim always imply trim and co-trim, but trim and co-trim do not imply eigenvalue trim and eigenvalue co-trim in general, respectively. In addition, the proposed eigenvalue trim reduction approach requires that the given Roesser model is not eigenvalue trim or not eigenvalue co-trim. Therefore, the reduction approaches for one block matrix in [23, 24, 32] are just some special cases of our new eigenvalue trim approach.

For the relationship between blocks, a transformation is given by Theorem 27 on page 119 in [23] as follows. Let an LFR be

$$\mathcal{F}(M, \Delta) = C\Delta(I_r - A\Delta)^{-1}B + D,$$

where $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathbf{R}^{(r+p)\times(r+q)}$ and $\Delta = Z = \mathrm{diag}\{z_1 I_{r_1}, \ldots, z_n I_{r_n}\}$. Suppose that $1 \leq \xi, \eta \leq r$ are such that

$$\begin{aligned} & a_{\xi,j} = 0 \quad \text{for all } j \neq \eta; \\ \text{and} \quad & a_{k,\eta} = 0 \quad \text{for all } k \neq \xi; \\ \text{and} \quad & b_{\xi,j} = 0, \quad \text{for all } j; \\ \text{and} \quad & c_{k,\eta} = 0 \quad \text{for all } k. \end{aligned} \tag{4.100}$$

Then, we have

$$C\Delta(I - A\Delta)^{-1}B = C\Pi_{\xi,\eta}\Delta(I - \Pi_{\xi,\eta}A\Pi_{\xi,\eta}\Delta)^{-1}\Pi_{\xi,\eta}B. \tag{4.101}$$

It should be noted that the conditions in (4.100) require that the entry $a_{\xi,\xi}$ in the given Roesser model must be zero. However, for our new approach, the conditions in (4.76) of Theorem 4.2 does not require $a_{\xi,\xi} = 0$, and if the given Roesser model meets the conditions (4.100), Theorem 4.2 will give the same result as (4.101). Therefore, the transformation given in [23] is just a special case of Theorem 4.2, or Theorem 4.2 can be viewed as a significant extension of the transformation given in [23].

Moreover, it should be noted that the 2-D Roesser model in (4.75) can be reduced to a lower-order Roesser model $(\hat{A}, \hat{B}, \hat{C}, D; \boldsymbol{r})$ in (4.99), whereas it cannot be reduced by the elementary operation reduction approach [32] and the n-D Jordan transformation approach [29].

Based on the above discussions, we see that the eigenvalue trim approach proposed in this paper is more general and effective than the existing methods given in [23, 24, 29, 32].

4.4.2 Application Examples

Example 4.6. *Hybrid dynamical systems are generally defined as dynamical systems where discrete and continuous dynamics interaction is involved. It can be found from biological systems, mechanical, electrical, electronic, chemical and industrial [72, 73]. Consider the 2-D hybrid system described by*

$$\begin{bmatrix} \dot{x}_1(t,i) \\ x_2(t,i+1) \end{bmatrix} = \begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{bmatrix} \begin{bmatrix} x_1(t,i) \\ x_2(t,i) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(t,i), \tag{4.102a}$$

$$y(t,i) = \begin{bmatrix} B_1 & B_2 \end{bmatrix} \begin{bmatrix} x_1(t,i) \\ x_2(t,i) \end{bmatrix} + Du(t,i), \tag{4.102b}$$

where

$$A = \begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{bmatrix} = \left[\begin{array}{cc|cccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 2 \end{array}\right], \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \left[\begin{array}{cc} 1 & 0 \\ 0 & 0 \\ \hline 4 & 3 \\ 2 & 3 \\ 4 & 4 \\ 6 & 5 \end{array}\right], \tag{4.103}$$

$$C = \begin{bmatrix} C_1 & C_2 \end{bmatrix} = \left[\begin{array}{cc|cccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right], \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \boldsymbol{r} = (2,\ 4),$$

which is used in [74] and is in fact a 2-D Roesser model.

For this model, the distinct eigenvalues of $A_{1,1}$ *and* $A_{2,2}$ *are* $\{1\}$, $\{1+\sqrt{2}, 1-\sqrt{2}\}$ *respectively. Note that for the eigenvalue* 1*, the matrix*

$$\begin{aligned} \tilde{M}_{1-} &= \begin{bmatrix} (A_{1,1} - 1I_2) & A_{1,2} & B_1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{aligned} \tag{4.104}$$

is not full row rank. Thus the model in (4.103) *is not eigenvalue trim and can be reduced. By utilizing the proposed eigenvalue order reduction approach, we can obtain a new 2-D* *hybrid system with order* 5 *or* $(r_1, r_2) = (1, 4)$ *as shown follows:*

$$\hat{A} = \begin{bmatrix} \hat{A}_{1,1} & \hat{A}_{1,2} \\ \hat{A}_{2,1} & \hat{A}_{2,2} \end{bmatrix} = \left[\begin{array}{c|cccc} 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 2 \end{array}\right], \quad \hat{B} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \left[\begin{array}{cc} 1 & 0 \\ \hline 4 & 3 \\ 2 & 3 \\ 4 & 4 \\ 6 & 5 \end{array}\right],$$

$$\hat{C} = \begin{bmatrix} C_1 & C_2 \end{bmatrix} = \left[\begin{array}{c|cccc} 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right], \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \boldsymbol{r} = (1,\ 4). \tag{4.105}$$

It should be noted that this model cannot be reduced by the trim approach given in [23] and elementary operation approach given in [29].

Example 4.7. *Consider the well-known example of* $a_{29}(z_1, z_2, z_3, z_4)$ *used in [3, 11, 19, 32], which is in fact the most complicated entry of the parametric system matrix* $A(\delta)$ *of the Research Civil Aircraft Model (RCAM) (see, e.g., [3, 75]),*

$$a_{29}(z_1, z_2, z_3, z_4) = \frac{\hat{a}_{29}(z_1, z_2, z_3, z_4)}{z_1 z_4^3} \tag{4.106}$$

with

$$\begin{aligned} \hat{a}_{29}(z_1, z_2, z_3, z_4) = & c_0 z_4^4 + c_1 z_1^2 z_2 z_3 + c_2 z_1 z_2^2 z_4^2 + c_3 z_1 z_2 z_3 z_4^2 + c_4 z_1^2 z_2^2 z_3 + c_5 z_2^2 z_3 z_4^4 \\ & + c_6 z_1 z_2^2 z_3 z_4^2 + c_7 z_1 z_2 z_4^2 + c_8 z_1^2 z_2 + c_9 z_1^2 + c_{10} z_2^2 z_4^2 + c_{11} z_1 z_4^2 \\ & + c_{12} z_2 z_4^4 + c_{13} z_3 z_4^4 + c_{14} z_1^2 z_3 + c_{15} z_2 z_3 z_4^4 + c_{16} z_1 z_3 z_4^2 + c_{17} z_1^2 z_2^2, \end{aligned}$$

where z_1 *is the mass;* z_2 *and* z_3 *are the two components of the position of center of gravity; and* z_3 *is the trimmed air speed, and* $c_0, c_1, \ldots, c_{17}$ *are the corresponding coefficients.*

It has been noted that $a_{29}(z_1, z_2, z_3, z_4)$ *is not causal and thus no standard or regular LFR (or Roesser model) realization can be directly found for it [11]. Although it is possible to convert* $a_{29}(z_1, z_2, z_3, z_4)$ *into a causal one by performing a normalization to uncertainties, this normalization should be avoided at such an early stage because it may increase the complexity of the problem and also the resultant LFR order [11, 33]. Therefore, instead of* $a_{29}(z_1, z_2, z_3, z_4)$*, one could consider first the Roesser model of*

$$H(z_1, z_2, z_3, z_4) \triangleq \begin{bmatrix} \hat{a}_{29}(z_1, z_2, z_3, z_4) \\ z_1 z_4^3 \end{bmatrix}.$$

By first applying the direct-construction approach [11] to it leads to a Roesser model (or an LFR) as

$$A = \left[\begin{array}{ccc|cccccc|c|ccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
\frac{c_{16}}{c_5} & \frac{c_{14}}{c_5} & 0 & 1 & \frac{c_{15}}{c_5} & \frac{c_5}{c_6} & \frac{c_3}{c_5} & \frac{c_4}{c_5} & \frac{c_1}{c_5} & 0 & \frac{c_{13}}{c_5} & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right], \quad B = \left[\begin{array}{c}
0 \\ 0 \\ 1 \\ \hline 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \hline 0 \\ \hline 0 \\ 0 \\ 0 \\ 0 \\ 1
\end{array}\right], \tag{4.107}$$

$$C = \left[\begin{array}{ccc|cccccc|c|ccccc}
c_{11} & c_9 & 0 & c_{10} & c_{12} & c_2 & c_7 & c_{17} & c_8 & c_5 & c_0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0
\end{array}\right], \quad D = \left[\begin{array}{c} 0 \\ 0 \end{array}\right],$$

with order 15, or more explicitly, $(r_1, r_2, r_3, r_4) = (3, 6, 1, 5)$.

For this model, distinct eigenvalues of $A_{2,2}$ *are 0 and the corresponding matrix* $\tilde{M}_{-2}$ *is not full column rank. Thus, this model is not eigenvalue co-trim, and can be reduced by the proposed approach. By the proposed eigenvalue trim approach, we can finally obtain a new lower-order Roesser model (or LFR) as*

$$A = \left[\begin{array}{ccc|cccc|c|ccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & a_{4,9} & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{5,9} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & a_{6,9} & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & a_{7,9} & 0 & 0 & 0 & 0 \\
\hline
\frac{c_{16}}{c_5} & \frac{c_{14}}{c_5} & 0 & \frac{c_5}{c_6} & \frac{c_3}{c_5} & \frac{c_4}{c_5} & \frac{c_1}{c_5} & 0 & \frac{c_{13}}{c_5} & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right], \quad B = \left[\begin{array}{c}
0 \\ 0 \\ 1 \\ \hline 0 \\ 0 \\ 0 \\ 0 \\ \hline 0 \\ \hline 0 \\ 0 \\ 0 \\ 0 \\ 1
\end{array}\right], \tag{4.108}$$

$$C = \left[\begin{array}{ccc|cccc|c|ccccc}
c_{11} & c_9 & 0 & c_2 & c_7 & c_{17} & c_8 & c_5 & c_0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0
\end{array}\right], \quad D = \left[\begin{array}{c} 0 \\ 0 \end{array}\right],$$

of order 13, or more explicitly, $(r_1, r_2, r_3, r_4) = (3, 4, 1, 5)$*, where*

$$\begin{aligned}
a_{4,9} &= \frac{c_6\left(c_5{}^2c_{15}c_{17}{}^2+c_2c_4{}^2c_6c_{12}+c_4c_5{}^2c_8c_{10}-c_4{}^2c_6c_7c_{10}-c_3c_5c_6c_{17}{}^2-c_1c_5{}^2c_{10}c_{17}-c_4c_5{}^2c_{12}c_{17}\right)}{(c_5{}^2c_{17}-c_2c_4c_6)^2} \\
&\quad+\frac{c_6(-c_2c_4c_5c_6c_8+c_1c_2c_5c_6c_{17}+c_4c_5c_6c_7c_{17}+c_3c_4c_6c_{10}c_{17}-c_2c_4c_6c_{15}c_{17})}{(c_5{}^2c_{17}-c_2c_4c_6)^2}, \\
a_{5,9} &= -\frac{c_6(c_4c_{10}-c_5c_{17})}{c_5{}^2c_{17}-c_2c_4c_6}, \\
a_{6,9} &= -\frac{c_5{}^4c_8c_{10}-c_5{}^4c_{12}c_{17}-c_2c_5{}^3c_6c_8+c_5{}^3c_6c_7c_{17}+c_1c_2{}^2c_5c_6{}^2-c_2{}^2c_4c_6{}^2c_{15}-c_1c_2c_5{}^2c_6c_{10}}{(c_5{}^2c_{17}-c_2c_4c_6)^2} \\
&\quad-\frac{c_2c_3c_4c_6{}^2c_{10}+c_2c_4c_5{}^2c_6c_{12}-c_4c_5{}^2c_6c_7c_{10}-c_2c_3c_5c_6{}^2c_{17}+c_2c_5{}^2c_6c_{15}c_{17}}{(c_5{}^2c_{17}-c_2c_4c_6)^2}, \\
a_{7,9} &= -\frac{c_5(c_2c_6-c_5c_{10})}{c_5{}^2c_{17}-c_2c_4c_6}.
\end{aligned} \tag{4.109}$$

Remark 4.9. *The 4-D Roesser model (4.107) can also be reduced to the Roesser model in (4.108) by the trim approach in [23], This is because for the 4-D Roesser model (4.107) all the eigenvalues of* $A_{1,1}$*,* $A_{2,2}$*,* $A_{3,3}$*,* $A_{4,4}$ *are 0, and the reduction condition for trim approach and eigenvalue trim are equivalent for such case.*

Remark 4.10. *It should be noted that once a lower-order* n*-D Roesser model of a given* n*-D filter or system is obtained by the proposed order reduction approach, it is straightforward to get a corresponding circuit implementation by the well-known techniques (see, e.g., [54, 76]), and thus such details are omitted.*

4.5 Contribution Summary

The notion of eigenvalue trim and eigenvalue co-trim for n-D Roesser model has been introduced, which reveals the internal connection between the eigenvalues and the reducibility of the considered Roesser model. Based on these results, sufficient conditions for reducibility and the corresponding order reduction algorithms for n-D Roesser model have been developed, which can achieve further order reduction than the existing approaches. Furthermore, a new transformation for n-D Roesser models, by swapping certain rows and columns and interchanging certain entries that belong to different blocks corresponding to different variables, has be established, which can transform an n-D Roesser model whose order cannot be reduced any more by the proposed approach to another equivalent Roesser model with the same order so that this transformed Roesser model can still be reduced further. Examples have been given to illustrate the details and the effectiveness of the new proposed approach.

Chapter 5

Common Eigenvector Approach to Exact Order Reduction for State-space Models of Multidimensional Systems

In the previous chapter, an eigenvalue trim approach has been proposed, where a preliminary relationship between eigenvalues and sufficient reducibility conditions of n-D Roesser model has been established. As mentioned in the introduction, the n-D models have a complex structure involving n different variables, which leads the n-D F-M model has n-D state matrix $A_1, \ldots, A_n$ and the state matrix of n-D Roesser is the block form. However, the limitation of the eigenvalue trim approach is that only part of the eigenvalues of the state matrix, i.e., only the eigenvalues of one block, are treated, and the task of giving a full exploration on the reducibility of n-D models by simultaneously taking into account of the eigenvalues of all the state matrices of the F-M models and blocks in the state matrix of Roesser model still remains unsolved.

In view of the above background, the purpose of this chapter is to establish a new n-D exact reduction approach where the eigenvalues of all the state matrices of the F-M models and blocks in the state-matrix of Roesser model will be simultaneously considered. Specifically, the notion of constrained common eigenvectors is introduced, for the first time, which provides insight into the relationship between reducibility and multiple eigenvalues. Based on this result, new reducibility conditions and the corresponding reduction procedure are developed for the F-M models, which make it possible to deal with eigenvalues of the state matrices $A_1, \ldots, A_n$, simultaneously. Then, these results are generalized to the

Roesser model, and it will be shown that this common eigenvector approach is applicable to a larger class of Roesser models for which the existing approaches may not be applied to do further order reduction. A Gröbner basis approach is proposed to compute such a constrained common eigenvector, which also leads to an equivalent reducibility condition. Moreover, a generalization to the state delay case is also given so that the eigenvalues of both the state matrix and the state-delay state matrix can be treated simultaneously.

The structure of this Chapter is as follows. In Section 5.1, sufficient reducibility conditions based on constrained common eigenvectors are developed for n-D F-M models, and then a corresponding reduction procedure is given. Section 5.2 generalize these results to the n-D Roesser model. In Section 5.3, a Gröbner basis method is established for the calculation of constrained common eigenvectors. Finally, conclusions are given in Section 5.4.

Following the notion of constrained eigenvector in [77], we first introduce the notion of the constrained common eigenvector, which will be used in this chapter.

Definition 5.1. *Let $A_1, \ldots, A_n \in \mathbf{R}^{r\times r}$, $B_1, \ldots, B_m \in \mathbf{R}^{r\times q}$, $C_1, \ldots, C_m \in \mathbf{R}^{p\times r}$. If a common right eigenvector $\boldsymbol{\omega}$ of $A_1, \ldots, A_n$ satisfies*

$$C_1\boldsymbol{\omega} = \ldots = C_m\boldsymbol{\omega} = \mathbf{0},$$

then it is said to be a common right eigenvector of $A_1, \ldots, A_n$ constrained by $C_1, \ldots, C_m$. Dually, if a common left eigenvector $\boldsymbol{\omega}$ of $A_1, \ldots, A_n$ satisfies

$$\boldsymbol{\omega}^{\mathrm{T}} B_1 = \ldots = \boldsymbol{\omega}^{\mathrm{T}} B_m = \mathbf{0},$$

then it is said to be a common left eigenvector of $A_1, \ldots, A_n$ constrained by $B_1, \ldots, B_m$.

For simplicity, such a vector $\boldsymbol{\omega}$ will just be referred to as a constrained common right/left eigenvector in the case of having no necessity to show explicitly the related matrices.

5.1 Reduction of F-M Models with Constrained Common Eigenvectors

This section is to develop conditions and the corresponding procedure for exact order reduction of the n-D F-M model by using the constrained common eigenvectors.

5.1.1 Reducibility Based on Constrained Common Right Eigenvectors

In this subsection, the exact reduction conditions about matrices $A_1, \ldots, A_n$ and C are developed by making use of the constrained common right eigenvectors. Then, a corresponding procedure is presented for the exact order reduction of the n-D F-M model.

Theorem 5.1. *For a given n-D F-M model $(\boldsymbol{A}, \boldsymbol{B}, C, D; r)$, if the system matrices A_1, $\ldots$, A_n have a common right eigenvector constrained by C, then the given n-D F-M model can be exactly reduced.*

Proof. Suppose that $A_1, \ldots, A_n$ have a common right eigenvector $\boldsymbol{\omega}$ constrained by C. Then, for every $i \in \{1, \ldots, n\}$ there exists an eigenvalue λ_i of A_i such that

$$A_i\,\boldsymbol{\omega} = \lambda_i\,\boldsymbol{\omega} \tag{5.1}$$

and

$$C\,\boldsymbol{\omega} = \mathbf{0}. \tag{5.2}$$

We can express this eigenvalue λ_i and the common eigenvector $\boldsymbol{\omega}$ as

$$\lambda_i = \alpha_i + j\beta_i, \quad \boldsymbol{\omega} = \boldsymbol{\mu} + j\boldsymbol{\nu}, \tag{5.3}$$

where j denotes the imaginary unit, the numbers α_i and β_i are the real part and the imaginary part, respectively, and $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ the corresponding vector ones. Note that if the eigenvector $\boldsymbol{\omega}$ corresponds to a real eigenvalue λ_i, $i \in \{1, \ldots, n\}$, the imaginary parts β_i, $i \in \{1, \ldots, n\}$, and $\boldsymbol{\nu}$ are both zero, i.e.,

$$\lambda_i = \alpha_i, \quad \boldsymbol{\omega} = \boldsymbol{\mu}. \tag{5.4}$$

Substituting (5.3) into (5.1) gives

$$A_i\boldsymbol{\mu} + jA_i\boldsymbol{\nu} = (\alpha_i\boldsymbol{\mu} - \beta_i\boldsymbol{\nu}) + j\,(\alpha_i\boldsymbol{\nu} + \beta_i\boldsymbol{\mu})\,, \tag{5.5}$$

which leads to

$$A_i\boldsymbol{\mu} = \alpha_i\boldsymbol{\mu} - \beta_i\boldsymbol{\nu}, \quad A_i\boldsymbol{\nu} = \alpha_i\boldsymbol{\nu} + \beta_i\boldsymbol{\mu}. \tag{5.6}$$

Similarly, Substituting the second equation in (5.3) into (5.2) gives

$$C\boldsymbol{\mu} + jC\boldsymbol{\nu} = \mathbf{0}, \tag{5.7}$$

and then

$$C\boldsymbol{\mu} = \mathbf{0}, \quad C\boldsymbol{\nu} = \mathbf{0}. \tag{5.8}$$

There exists a full column rank matrix

$$R \triangleq \begin{bmatrix} \boldsymbol{t}_1 & \cdots & \boldsymbol{t}_{r-2} \end{bmatrix} \in \mathbf{R}^{r\times\hat{r}} \tag{5.9}$$

which completes the matrix

$$\tilde{R} \triangleq \begin{bmatrix} \boldsymbol{\mu} & \boldsymbol{\nu} \end{bmatrix} \in \mathbf{R}^{r\times\tilde{r}} \tag{5.10}$$

to a nonsingular matrix $T \in \mathbf{R}^{r\times r}$ as

$$T = \left[\begin{array}{c|c} \tilde{R} & R \end{array}\right] = \left[\begin{array}{cc|ccc} \boldsymbol{\mu} & \boldsymbol{\nu} & \boldsymbol{t}_1 & \cdots & \boldsymbol{t}_{r-2} \end{array}\right]. \tag{5.11}$$

Partition T^{-1} as

$$T^{-1} \triangleq \left[\begin{array}{c} \tilde{L} \\ \hline L \end{array}\right], \tag{5.12}$$

with $\tilde{L} \in \mathbf{R}^{\times\tilde{r}\times r}$ and $L \in \mathbf{R}^{\tilde{r}\times r}$.

Since one can see from (5.6) that for every $i \in \{1,\ldots,n\}$, $A_i\boldsymbol{\mu}$ and $A_i\boldsymbol{\nu}$ can be expressed as a linear combination of the vectors $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$, and for every $i \in \{1,\ldots,n\}$ and $k \in \{1,\ldots,\hat{r}\}$, $A_i\boldsymbol{t_k}$ can be expressed as a linear combination of the column vectors in T, we have that

$$\begin{aligned} A_iT &= \left[\begin{array}{cc|ccc} A_i\boldsymbol{\mu} & A_i\boldsymbol{\nu} & A_i\boldsymbol{t}_1 & \cdots & A_i\boldsymbol{t}_{r-2} \end{array}\right] \\ &= \left[\begin{array}{cc|ccc} \boldsymbol{\mu} & \boldsymbol{\nu} & \boldsymbol{t}_1 & \cdots & \boldsymbol{t}_{r-2} \end{array}\right] \left[\begin{array}{cc|ccc} \star & \star & \star & & \star \\ \star & \star & \star & & \star \\ \hline 0 & 0 & \star & & \star \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \star & & \star \end{array}\right] \\ &\triangleq T \left[\begin{array}{c|c} \tilde{A}_i & \check{A}_i \\ \hline \mathbf{0} & \hat{A}_i \end{array}\right], \end{aligned} \tag{5.13}$$

with $\star$ being some real constant numbers.

Pre-multiplying (5.13) by T^{-1} gives

$$\left[\begin{array}{c|c} \tilde{A}_i & \check{A}_i \\ \mathbf{0} & \hat{A}_i \end{array}\right] = T^{-1}A_iT = \left[\begin{array}{c} \tilde{L} \\ \hline L \end{array}\right] A_i \left[\begin{array}{c|c} \tilde{R} & R \end{array}\right] = \left[\begin{array}{c|c} \tilde{L}A_i\tilde{R} & \tilde{L}A_iR \\ \hline LA_i\tilde{R} & LA_iR \end{array}\right]$$

$$= \left[\begin{array}{c|c} \tilde{L}A_i\tilde{R} & \tilde{L}A_iR \\ \hline \mathbf{0} & LA_iR \end{array}\right]. \tag{5.14}$$

It follows form (5.8) that

$$= CT = C\left[\begin{array}{c|c} \tilde{R} & R \end{array}\right] = \left[\begin{array}{c|c} \boldsymbol{R} & CR \end{array}\right]. \tag{5.15}$$

We see from (5.14) and (5.15) that

$$\begin{aligned}
&C(I_r - \sum_{i=1}^{n} z_iA_i)^{-1}\left(\sum_{i=1}^{n} z_iB_i\right) \\
=&CTT^{-1}\left(I_r - \sum_{i=1}^{n} z_iA_i\right)^{-1}\left(\sum_{i=1}^{n} z_iCTT^{-1}B_i\right) \\
=&CT\left(I_r - \sum_{i=1}^{n} z_iT^{-1}A_iT\right)^{-1}\left(\sum_{i=1}^{n} z_iCT^{-1}B_i\right) \\
=&\left[\begin{array}{cc} \mathbf{0} & CR \end{array}\right]\left[\begin{array}{cc} -\sum_{i=1}^{n} z_iLA_i\tilde{R} & -\sum_{i=1}^{n} z_i\tilde{L}A_iR \\ \mathbf{0} & I_{\hat{r}} - \sum_{i=1}^{n} z_iLA_iR \end{array}\right]^{-1}\left[\begin{array}{c} \sum_{i=1}^{n} z_i\tilde{L}B_i \\ \sum_{i=1}^{n} z_iLB_i \end{array}\right] \\
=&CR\left(I_{\hat{r}} - \sum_{i=1}^{n} z_iLA_iR\right)^{-1}\left(\sum_{i=1}^{n} z_iLB_i\right).
\end{aligned} \tag{5.16}$$

That is to say, a new n-D F-M model $(\hat{\boldsymbol{A}}, \hat{\boldsymbol{B}}, \hat{C}, \hat{D}; \hat{r})$ with the lower order $\hat{r}$ and $\hat{C} = CR$, $\hat{A}_i = LA_iR$, $\hat{B}_i = LB_i$, has been obtained. □

Remark 5.1. *It should be noted that by Theorem 5.1 an n-D F-M model is always reducible as long as the matrices $A_1, \ldots, A_n$ have a common right eigenvector $\boldsymbol{\omega}$ constrained by C. Therefore, one can apply again, if possible, Theorem 5.1 to the resultant F-M model, and repeat this process until no common constrained right eigenvector to achieve an n-D F-M model with a lowest possible order.*

A key point for applying Theorem 5.1 is how to find a constrained common right eigenvector. In the following, we give another equivalent theorem based on eigenvalues, which provides a method for finding such a constrained common right eigenvector.

Theorem 5.2. *For an n-D F-M model* $(\boldsymbol{A}, \boldsymbol{B}, C, D; r)$*, if the matrix*

$$F_{\mathrm{R}} \triangleq \begin{bmatrix} (A_1 - \lambda_1 I_r) \\ \vdots \\ (A_n - \lambda_n I_r) \\ C \end{bmatrix} \tag{5.17}$$

is not of full row rank for some λ_i *being an eigenvalue of* A_i*,* $i = 1, \ldots, n$*, then the given n-D F-M model can be exactly reduced.*

Proof. By the definition of constrained common right eigenvector, we have that matrices A_1, …, A_n have a common right eigenvector constrained by C if and only if there exists an eigenvalue λ_i of A_i for every $i \in \{1, \ldots, n\}$, such that

$$A_i \boldsymbol{\omega} = \lambda_i \boldsymbol{\omega}, \tag{5.18a}$$

$$C_i \boldsymbol{\omega} = \mathbf{0}. \tag{5.18b}$$

Equations in (5.17) is equivalent to

$$F_{\mathrm{R}} \boldsymbol{\omega} = 0. \tag{5.19}$$

with F_{L} in (5.17), which is equivalent to the rank deficient of the matrix F_{L} in (5.17). Thus, we have that A_1, …, A_n have a common right eigenvector $\boldsymbol{\omega}$ constrained by C if and only if the matrix F_{L} in (5.17) is not full column rank for some eigenvalues λ_1, …, λ_n of A_1, …, A_n, respectively. In view of Theorem 5.1, we can conclude Theorem 5.2. □

Remark 5.2. *It should be noted that the exact order reduction method proposed in [15, 29] can only treat a single eigenvalue. However, the condition of Theorem 5.2 is developed for multiple eigenvalues, i.e., eigenvalues* $\lambda_1, \ldots, \lambda_n$ *of* $A_1, \ldots, A_n$*, respectively.*

Remark 5.3. *We would like to remark that the proof of Theorem 5.2 indicates a method to obtain a common right eigenvector* $\boldsymbol{\omega}$ *of* $A_1, \ldots, A_n$ *constrained by* C*. First, select an eigenvalues* λ_1*, …,* λ_n *of* A_1*, …,* A_n*, respectively, such that the matrix* F_{R} *in* (5.17) *is not of full column rank. Second, find a nonzero vector* $\boldsymbol{\omega}$ *such that* $F_{\mathrm{R}}\boldsymbol{\omega} = \mathbf{0}$*. In order to establish a direct way without the knowledge of eigenvalues, another method based Gröbner basis will be given in Section 5.3.*

Now, a basic reduction procedure based on the so-called constrained common right eigenvector is presented as shown in Procedure 5.1 to achieve an n-D F-M model with order as low as possible.

Procedure 5.1: Exact Order Reduction of an n-D F-M Model Using a Constrained Common Right Eigenvector

---

Input : A given n-D F-M model $(\boldsymbol{A}, \boldsymbol{B}, C, D; r)$;
Output: A reduced-order n-D F-M model $(\hat{\boldsymbol{A}}, \hat{\boldsymbol{B}}, \hat{C}, \hat{D}; \hat{r})$;

while $A_1, \ldots, A_n$ *share a common right eigenvector* $\boldsymbol{\omega}$ *constrained by* C **do**

Step 1: Express $\boldsymbol{\omega}$ as

$$\boldsymbol{\omega} = \boldsymbol{\mu} + j\boldsymbol{\nu} \tag{5.20}$$

and construct a nonsingular matrix T in the form (5.11) ;

Step 2: Extract R and L from T of (5.11) and T^{-1} of (5.12), respectively;

Step 3: Obtain a reduced-order n-D F-M model $(\hat{\boldsymbol{A}}, \hat{\boldsymbol{B}}, \hat{C}, \hat{D}; \hat{r})$ by

$$\begin{aligned} \hat{A}_i =& LA_iR, \quad \hat{B}_i = LB_i, \quad i = 1, \ldots, n, \\ \hat{C} =& CR, \quad \hat{D} = D; \end{aligned} \tag{5.21}$$

Renew the n-D F-M model $(\boldsymbol{A}, \boldsymbol{B}, C, D; r)$ as $(\hat{\boldsymbol{A}}, \hat{\boldsymbol{B}}, \hat{C}, \hat{D}; \hat{r})$;

end

return *the reduced-order F-M model* $(\hat{\boldsymbol{A}}, \hat{\boldsymbol{B}}, \hat{C}, \hat{D}; \hat{r}) \triangleq (\boldsymbol{A}, \boldsymbol{B}, C, D; r)$.

The following example is given to show more details and effectiveness of the proposed reduction procedure based on constrained common right eigenvectors.

Example 5.1. *Consider the 2-D F-M model* $(\boldsymbol{A}, \boldsymbol{B}, C, D; r)$*:*

$$\begin{aligned} A_1 &= \begin{bmatrix} -2 & 1 & 0 & -1 & 0 \\ -4 & -3 & 0 & -1 & -1 \\ -5 & 5 & 1 & 3 & 1 \\ 9 & 3 & -1 & 1 & -1 \\ -2 & -4 & -1 & -3 & -2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -4 & 0 & 0 & -1 & -1 \\ -2 & -3 & 0 & 0 & 0 \\ -5 & 0 & 0 & 2 & 0 \\ 7 & 4 & -1 & -1 & -2 \\ -1 & -2 & 0 & -1 & 0 \end{bmatrix}, \\ B_1 &= \begin{bmatrix} 1 \\ 2 \\ 2 \\ -4 \\ 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 \\ -1 \\ -2 \\ 4 \\ -1 \end{bmatrix}, \\ C &= \begin{bmatrix} 1 & 2 & 1 & 2 & 3 \\ 2 & 4 & 2 & 4 & 5 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad r = 5. \end{aligned} \tag{5.22}$$

It can be verified that corresponding the eigenvalues $\lambda_1 = -1-3j$ *and* $\lambda_2 = -3-2j$ *of* A_1 *and* A_2*, respectively, the vector*

$$\boldsymbol{\omega} = \begin{bmatrix} 1 \\ -j \\ 1-2j \\ -1+2j \\ 0 \end{bmatrix} \tag{5.23}$$

is a common right eigenvector of A_1 *and* A_2 *constrained by* C*. Thus, the F-M model in* (5.22) *can be reduced by applying Procedure 5.1. the specific reduction steps are as follows.*

Step 1: *Since the constrained common right eigenvector* $\boldsymbol{\omega}$ *is complex, it can be expressed as*

$$\begin{aligned} \boldsymbol{\omega} &= \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} + j \begin{bmatrix} 0 \\ -1 \\ -2 \\ 2 \\ 0 \end{bmatrix} \\ &\triangleq \boldsymbol{\mu} + j\boldsymbol{\nu}. \end{aligned} \tag{5.24}$$

We can construct to a nonsingular matrix

$$\begin{aligned} T &= \left[\begin{array}{cc|ccc} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 \\ -1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right] \\ &\triangleq \left[\begin{array}{c|c} \tilde{R} & R \end{array}\right]. \end{aligned} \tag{5.25}$$

Step 2: *We extract the matrix* L *from* T^{-1}*:*

$$\begin{aligned} T^{-1} &= \left[\begin{array}{ccccc} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ \hline -1 & -2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right] \\ &\triangleq \left[\begin{array}{c} \tilde{L} \\ \hline L \end{array}\right]. \end{aligned} \tag{5.26}$$

Step 3: *We then obtain a new F-M model:*

$$
\begin{aligned}
\hat{A}_1 =& LA_1R = \begin{bmatrix} 1 & 6 & 3 \\ -1 & -2 & -3 \\ -1 & -3 & -2 \end{bmatrix}, \\
\hat{A}_2 =& LA_2R = \begin{bmatrix} 0 & 3 & 1 \\ -1 & -2 & -3 \\ 0 & -1 & 0 \end{bmatrix}, \\
\hat{B}_1 =& LB_1 = \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}, \\
\hat{B}_2 =& LB_2 = \begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix}, \\
\hat{C} =& CR = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \end{bmatrix}, \\
\hat{D} =& D = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.
\end{aligned}
\tag{5.27}
$$

It is seen that the order of the obtained 2-D F-M model (5.27) is $\hat{r} = 3$, which is lower than that of 5 for the given one in (5.22).

5.1.2 Reducibility Based on Constrained Common Left Eigenvectors

In the previous subsection, the relationship of reducibility to the matrices $A_1, \ldots, A_n$ and C has been revealed based on the constrained common right eigenvectors. In this subsection, we will further clarify, based on the constrained common left eigenvectors, the relationship of the reducibility of the F-M model to the matrices A_1, ..., A_n and B_1, ..., B_n.

It should be noted, however, that different to the conventional 1-D model and the n-D Roesser model, the duality between an n-D F-M model and its transpose does not hold [37, 78]. Thus, the results obtained for $A_1, \ldots, A_n$ and $B_1, \ldots, B_n$ cannot be directly applied to the case for $A_1, \ldots, A_n$ and C via the transpose of the given F-M model, and some alternative method or special treatment has to be adopted. It will be shown in the following that, by introducing a notion called pseudo duality for the F-M model, we can establish the desired relationship for $A_1, \ldots, A_n$ and C, by utilizing the results obtained in Theorems 5.1 and 5.2.

Definition 5.2. *For a given n-D F-M model* $(\boldsymbol{A}, \boldsymbol{B}, C, D; r)$*, its pseudo dual F-M model* $(\boldsymbol{A}_{\mathrm{D}}, \boldsymbol{B}_{\mathrm{D}}, C_{\mathrm{D}}, D_{\mathrm{D}}; r)$ *is defined as*

$$
\begin{aligned}
A_{\mathrm{D}} &= \begin{pmatrix} A_{\mathrm{D}1} & \dots & A_{\mathrm{D}n} \end{pmatrix} \triangleq (A_1^{\mathrm{T}}, \dots, A_n^{\mathrm{T}}), \\
B_{\mathrm{D}} &= \begin{pmatrix} B_{\mathrm{D}1} & B_{\mathrm{D}2} & \dots & B_{\mathrm{D}n} \end{pmatrix} \triangleq \begin{pmatrix} C^{\mathrm{T}} & \mathbf{0} & \dots & \mathbf{0} \end{pmatrix}, \\
C_{\mathrm{D}} &= \begin{bmatrix} C_{\mathrm{D}1} \\ \vdots \\ C_{\mathrm{D}n} \end{bmatrix} \triangleq \begin{bmatrix} B_1^{\mathrm{T}} \\ \vdots \\ B_n^{\mathrm{T}} \end{bmatrix}, \qquad (5.28)
\end{aligned}
$$

$$
D_{\mathrm{D}} \triangleq \begin{bmatrix} D^{\mathrm{T}} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix}. \qquad (5.29)
$$

Then, reduction conditions based on constrained common left eigenvectors and eigenvalues are developed as follows.

Theorem 5.3. *For an n-D F-M model* $(\boldsymbol{A}, \boldsymbol{B}, C, D; r)$*, if matrices* $A_1, \dots, A_n$ *have a common left eigenvector* $\boldsymbol{\omega}$ *constrained by* $B_1, \dots, B_n$ *then the given n-D F-M model* $(\boldsymbol{A}, \boldsymbol{B}, C, D; r)$ *can be exactly reduced.*

Proof. Let $(\boldsymbol{A}_{\mathrm{D}}, \boldsymbol{B}_{\mathrm{D}}, C_{\mathrm{D}}, D_{\mathrm{D}}; r)$ be the pseudo dual F-M model of $(\boldsymbol{A}, \boldsymbol{B}, C, D; r)$. We then have that

$$
C_{\mathrm{D}} \left(I_r - \sum_{i=1}^{n} z_i A_{\mathrm{D}i} \right)^{-1} \sum_{i=1}^{n} z_i B_{\mathrm{D}i} \qquad (5.30)
$$

$$
\begin{aligned}
&= \begin{bmatrix} B_1^{\mathrm{T}} \\ \vdots \\ B_n^{\mathrm{T}} \end{bmatrix} \left(I_r - \sum_{i=1}^{n} z_i A_i^{\mathrm{T}} \right)^{-1} z_1 C^{\mathrm{T}} \\
&= \begin{bmatrix} B_1^{\mathrm{T}} \left(I_r - \sum_{i=1}^{n} z_i A_i^{\mathrm{T}} \right)^{-1} z_1 C^{\mathrm{T}} \\ \vdots \\ B_n^{\mathrm{T}} \left(I_r - \sum_{i=1}^{n} z_i A_i^{\mathrm{T}} \right)^{-1} z_1 C^{\mathrm{T}} \end{bmatrix}. \qquad (5.31)
\end{aligned}
$$

Since $A_1, \dots, A_n$ have a common left eigenvector $\boldsymbol{\omega}$ constrained by B_1, …, B_n, by the duality of the constrained common eigenvector we then have that $\boldsymbol{\omega}$ is a common right eigenvector of $A_{\mathrm{D}1}, \ldots, A_{\mathrm{D}n}$ constrained by C_{D}. Then, it follows from Theorem 5.1 and Procedure 5.1 that the F-M model $(\boldsymbol{A}_{\mathrm{D}}, \boldsymbol{B}_{\mathrm{D}}, C_{\mathrm{D}}, D_{\mathrm{D}}; r)$ can be reduced to a new F-M model, say $(\hat{\boldsymbol{A}}_{\mathrm{D}}, \hat{\boldsymbol{B}}_{\mathrm{D}}, \hat{C}_{\mathrm{D}}, \hat{D}_{\mathrm{D}}; \hat{r})$ with $\hat{r} < r$, which is obtained by

$$\begin{aligned}
\hat{\mathbf{A}}_{\mathrm{D}} &= \left(\hat{A}_{\mathrm{D}1}, \ldots, \hat{A}_{\mathrm{D}n}\right) \triangleq (LA_{\mathrm{D}1}R, \ldots, LA_{\mathrm{D}n}R) = (LA_1^{\mathrm{T}}R, \ldots, LA_n^{\mathrm{T}}R), \\
\hat{\mathbf{B}}_{\mathrm{D}} &= \left(\hat{B}_{\mathrm{D}1}, \hat{B}_{\mathrm{D}2}, \ldots, \hat{B}_{\mathrm{D}n}\right) \triangleq (LB_{\mathrm{D}1}, LB_{\mathrm{D}2}, \ldots, LB_{\mathrm{D}n}) \triangleq \left(LC^{\mathrm{T}}, \mathbf{0}, \ldots, \mathbf{0}\right), \\
\hat{C}_{\mathrm{D}} &= \begin{bmatrix} \hat{C}_{\mathrm{D}1} \\ \vdots \\ \hat{C}_{\mathrm{D}n} \end{bmatrix} \triangleq \begin{bmatrix} C_{\mathrm{D}1} \\ \vdots \\ C_{\mathrm{D}n} \end{bmatrix} R = \begin{bmatrix} B_1^{\mathrm{T}} \\ \vdots \\ B_n^{\mathrm{T}} \end{bmatrix} R = \begin{bmatrix} B_1^{\mathrm{T}}R \\ \vdots \\ B_n^{\mathrm{T}}R \end{bmatrix}, \\
\hat{D}_{\mathrm{D}} &\triangleq D_{\mathrm{D}} = [D \ \mathbf{0} \ \ldots \ \mathbf{0}]^{\mathrm{T}},
\end{aligned} \tag{5.32}$$

with appropriate real matrices L and R. We then find that

$$\begin{aligned}
& C_{\mathrm{D}} \left(I_r - \sum_{i=1}^{n} z_i A_{\mathrm{D}i}\right)^{-1} \left(\sum_{i=1}^{n} z_i B_{\mathrm{D}i}\right) \\
=& \hat{C}_{\mathrm{D}} \left(I_{\hat{r}} - \sum_{i=1}^{n} z_i \hat{A}_{\mathrm{D}i}\right)^{-1} \left(\sum_{i=1}^{n} z_i \hat{B}_{\mathrm{D}i}\right) \\
=& \begin{bmatrix} \hat{C}_{\mathrm{D}1} \\ \vdots \\ \hat{C}_{\mathrm{D}n} \end{bmatrix} \left(I_{\hat{r}} - \sum_{i=1}^{n} z_i \hat{A}_{\mathrm{D}i}\right)^{-1} z_1 \hat{B}_{\mathrm{D}1} \\
=& \begin{bmatrix} \hat{C}_{\mathrm{D}1} \left(I_{\hat{r}} - \sum_{i=1}^{n} z_i \hat{A}_{\mathrm{D}i}\right)^{-1} z_1 \hat{B}_{\mathrm{D}1} \\ \vdots \\ \hat{C}_{\mathrm{D}n} \left(I_{\hat{r}} - \sum_{i=1}^{n} z_i \hat{A}_{\mathrm{D}i}\right)^{-1} z_1 \hat{B}_{\mathrm{D}1} \end{bmatrix}.
\end{aligned} \tag{5.33}$$

It follows from From (5.30) and (5.33) that

$$B_k^{\mathrm{T}} \left(I_r - \sum_{i=1}^{n} z_i A_i^{\mathrm{T}}\right)^{-1} z_1 C^{\mathrm{T}} = C_{\mathrm{D}k} \left(I_{\hat{r}} - \sum_{i=1}^{n} z_i \hat{A}_{\mathrm{D}i}\right)^{-1} z_1 \hat{B}_{\mathrm{D}1} \tag{5.34}$$

and then

$$B_k^{\mathrm{T}} \left(I_r - \sum_{i=1}^{n} z_i A_i^{\mathrm{T}}\right)^{-1} C^{\mathrm{T}} = C_{\mathrm{D}k} \left(I_{\hat{r}} - \sum_{i=1}^{n} z_i \hat{A}_{\mathrm{D}i}\right)^{-1} \hat{B}_{\mathrm{D}1}, \tag{5.35}$$

for every $k \in \{1, \ldots, n\}$. Transposing (5.35) gives

$$C\left(I_r - \sum_{i=1}^{n} z_i A_i\right)^{-1} B_k = \hat{B}_{\mathrm{D}1}^{\mathrm{T}} \left(I_{\hat{r}} - \sum_{i=1}^{n} z_i \hat{A}_{\mathrm{D}i}^{\mathrm{T}}\right)^{-1} C_{\mathrm{D}k}^{\mathrm{T}}, \tag{5.36}$$

for every $k \in \{1, \ldots, n\}$. Therefore,

$$\begin{aligned} & C\left(I_r - \sum_{i=1}^{n} z_i A_i\right)^{-1} \sum_{i=1}^{n} z_i B_i \\ =& \hat{B}_{\mathrm{D}1}^{\mathrm{T}} \left(I_{\hat{r}} - \sum_{i=1}^{n} z_i \hat{A}_{\mathrm{D}i}^{\mathrm{T}}\right)^{-1} \sum_{i=1}^{n} z_i C_{\mathrm{D}i}^{\mathrm{T}} \\ \triangleq& \hat{C}\left(I_r - \sum_{i=1}^{n} z_i \hat{A}_i\right)^{-1} \sum_{i=1}^{n} z_i \hat{B}_i, \end{aligned} \tag{5.37}$$

with

$$\hat{A}_i \triangleq \hat{A}_{\mathrm{D}i}^{\mathrm{T}}, \quad \hat{B}_i \triangleq \hat{C}_{\mathrm{D}i}^{\mathrm{T}}, \quad \hat{C} \triangleq \hat{B}_{\mathrm{D}1}^{\mathrm{T}}.$$

That is to say, the given n-D F-M model $(\boldsymbol{A}, \boldsymbol{B}, C, D; r)$ can be reduced to a new F-M model $(\hat{\boldsymbol{A}}, \hat{\boldsymbol{B}}, \hat{C}, \hat{D}; \hat{r})$ with

$$\hat{\boldsymbol{A}} = (\hat{A}_{\mathrm{D}1}^{\mathrm{T}}, \ldots, \hat{A}_{\mathrm{D}n}^{\mathrm{T}}), \quad \hat{\boldsymbol{B}} = (\hat{C}_{\mathrm{D}1}^{\mathrm{T}}, \ldots, \hat{C}_{\mathrm{D}n}^{\mathrm{T}}), \quad \hat{C} = \hat{B}_{\mathrm{D}1}^{\mathrm{T}}, \quad \hat{D} = D.$$

□

Theorem 5.4. *For an n-D F-M model $(\boldsymbol{A}, \boldsymbol{B}, C, D; r)$, if*

$$F_{\mathrm{L}} \triangleq \begin{bmatrix} (A_1 - \lambda_1 I_r) & \cdots & (A_n - \lambda_n I_r) & B_1 & \ldots & B_n \end{bmatrix}, \tag{5.38}$$

is not of full column rank for some λ_i being an eigenvalue of A_i, $i = 1, \ldots, n$, then the given n-D F-M model can be exactly reduced.

Proof. The proof can be done in a similar way to the one of Theorem 5.2. □

Remark 5.4. *From the proof of Theorem 5.3, we can find that an n-D F-M model $(\boldsymbol{A}, \boldsymbol{B}, C, D; r)$ satisfying the condition of Theorem 5.3 or Theorem 5.4 can be reduced as follows: Fist, apply Procedure 5.1 to reduce the pseudo dual F-M model $(\boldsymbol{A}_{\mathrm{D}}, \boldsymbol{B}_{\mathrm{D}}, C_{\mathrm{D}}, D_{\mathrm{D}}; r)$*

of the given one to get a lower-order model $(\hat{\boldsymbol{A}}_{\mathrm{D}}, \hat{\boldsymbol{B}}_{\mathrm{D}}, \hat{C}_{\mathrm{D}}, \hat{D}_{\mathrm{D}}; \hat{r})$ *with*

$$\hat{\mathbf{A}}_{\mathrm{D}} = (\hat{\mathbf{A}}_{\mathbf{D1}}, \ldots, \hat{\mathbf{A}}_{\mathbf{D}\boldsymbol{n}}), \quad \hat{\mathbf{B}}_{\mathrm{D}} = (\hat{\mathbf{B}}_{\mathbf{D1}}, \mathbf{0}, \ldots, \mathbf{0}), \quad \hat{\mathbf{C}}_{\mathrm{D}} = \begin{bmatrix} \hat{C}_{\mathrm{D1}} \\ \vdots \\ \hat{C}_{\mathrm{D}n} \end{bmatrix}, \quad \hat{\mathbf{D}}_{\mathrm{D}} = \begin{bmatrix} D^{\mathrm{T}} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix}.$$

Then, obtain a reduced-order F-M model $(\hat{\boldsymbol{A}}, \hat{\boldsymbol{B}}, \hat{C}, \hat{D}; \hat{r})$ *for the given one by setting* $\hat{\boldsymbol{A}} \triangleq (\hat{A}_{\mathrm{D1}}^{\mathrm{T}}, \ldots, \hat{A}_{\mathrm{D}n}^{\mathrm{T}})$*,* $\hat{\boldsymbol{B}} \triangleq (\hat{C}_{\mathrm{D1}}^{\mathrm{T}}, \ldots, \hat{C}_{\mathrm{D}n}^{\mathrm{T}})$*,* $\hat{C} \triangleq \hat{B}_{\mathrm{D1}}^{\mathrm{T}}$ *and* $\hat{D} \triangleq D$*.*

Example 5.2. *To see more details, let us consider the 2-D F-M model* $(\hat{\boldsymbol{A}}, \hat{\boldsymbol{B}}, \hat{C}, \hat{D}; \hat{r})$ *in (5.27) obtained from (5.39) in Example 5.2, and redefine* $(A, B, C, D; \boldsymbol{r})$ *as* $(\boldsymbol{A}, \boldsymbol{B}, C, D; r) \triangleq (\hat{\boldsymbol{A}}, \hat{\boldsymbol{B}}, \hat{C}, \hat{D}; \hat{r})$*. It can be checked that for the eigenvalues* $\lambda_1 = -2$ *and* $\lambda_2 = -1$ *of* A_1 *and* A_2*, respectively, the matrix*

$$F_{\mathrm{L}} = \begin{bmatrix} (A_1 - \lambda_1 I_3) & (A_2 - \lambda_2 I_3) & B_1 & B_2 \end{bmatrix} \tag{5.39}$$

has rank 2 and is not of full column rank. Thus, the given 2-D F-M model can be reduced.

The pseudo dual F-M model $(\boldsymbol{A}_{\mathrm{D}}, \boldsymbol{B}_{\mathrm{D}}, C_{\mathrm{D}}, D_{\mathrm{D}}; r)$ *of* $(\boldsymbol{A}, \boldsymbol{B}, C, D; r)$ *is:*

$$\begin{aligned} A_{\mathrm{D1}} &= \begin{bmatrix} 1 & -1 & -1 \\ 6 & -2 & -3 \\ 3 & -3 & -2 \end{bmatrix}, \quad A_{\mathrm{D2}} = \begin{bmatrix} 0 & -1 & 0 \\ 3 & -2 & -1 \\ 1 & -3 & 0 \end{bmatrix}, \\ B_{\mathrm{D1}} &= \left[\begin{array}{c|c} 1 & 2 \\ 2 & 4 \\ 3 & 5 \end{array}\right], \quad B_{\mathrm{D2}} = \mathbf{0}, \\ C_{\mathrm{D}} &= \left[\begin{array}{ccc} -3 & 1 & 1 \\ \hline -1 & 3 & -1 \end{array}\right], \quad D_{\mathrm{D}} = \left[\begin{array}{cc} 0 & 0 \\ \hline 0 & 0 \end{array}\right]. \end{aligned} \tag{5.40}$$

Then, applying Procedure 1 to it yields

$$\begin{aligned} \hat{A}_{\mathrm{D1}} &= \begin{bmatrix} -1 & -2 \\ -1 & 0 \end{bmatrix}, \quad \hat{A}_{\mathrm{D2}} = \begin{bmatrix} -1 & -1 \\ -1 & 0 \end{bmatrix}, \\ \hat{B}_{\mathrm{D1}} &= \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}, \quad \hat{B}_{\mathrm{D2}} = \mathbf{0}, \\ \hat{C}_{\mathrm{D}} &= \left[\begin{array}{cc} 1 & 1 \\ \hline 3 & -1 \end{array}\right], \quad \hat{D}_{\mathrm{D}} = \left[\begin{array}{cc} 0 & 0 \\ \hline 0 & 0 \end{array}\right], \quad \hat{r} = 2. \end{aligned} \tag{5.41}$$

Finally, we can obtain a reduced-order F-M model:

$$\begin{aligned}
&\hat{A}_1 \triangleq \hat{A}_{\mathrm{D}1}^{\mathrm{T}} = \begin{bmatrix} -1 & -1 \\ -2 & 0 \end{bmatrix}, \quad \hat{A}_2 \triangleq \hat{A}_{\mathrm{D}2}^{\mathrm{T}} = \begin{bmatrix} -1 & -1 \\ -1 & 0 \end{bmatrix}, \\
&\hat{B}_1 \triangleq \hat{C}_{\mathrm{D}1}^{\mathrm{T}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \hat{B}_2 \triangleq \hat{C}_{\mathrm{D}2}^{\mathrm{T}} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}, \quad \hat{C} = \hat{B}_{\mathrm{D}1}^{\mathrm{T}} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}, \\
&\hat{D} \triangleq D = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \hat{r} = 2.
\end{aligned} \tag{5.42}$$

In this way, the order of the 2-D F-M model given in (5.22) has been finally reduced to 2 from 5 by the proposed approach.

Example 5.3. *Consider again the 2-D F-M model in* (3.30). *For the eigenvalues* $\lambda_1 = 1$ *and* $\lambda_2 = 0$ *of* A_1 *and* A_2*, respectively, the matrix*

$$F_{\mathrm{R}} = \begin{bmatrix} (A_1 - \lambda_1 I_3) \\ (A_2 - \lambda_2 I_3) \\ C \end{bmatrix} = \left[\begin{array}{cc} 1 & 1 \\ 1 & 1 \\ \hline 1 & 1 \\ 1 & 1 \\ \hline 1 & 1 \end{array}\right]$$

has rank 1 *and is not of full rank, then the F-M model* (3.30) *can be reduced. By the proposed reduction algorithm and Remark 5.4, we can obtain the reduced-order F-M model* (3.35) *for the original F-M model* (3.30).

Remark 5.5. *Theorems 5.1-5.4 give sufficient conditions for exact order reduction of n-D F-M models based on common eigenvectors and eigenvalues. These results can be viewed as the generalization of the Popov-Belevitch-Hautus (PBH) tests for the reducibility of the the conventional 1-D case based on eigenvectors and eigenvalues.*

Remark 5.6. *For a given n-D F-M model* $(\boldsymbol{A}, \boldsymbol{B}, C, D; r)$*, the reduction conditions of Theorems 5.3 and 5.4 are as follows: The state matrices* $A_1, \ldots, A_n$ *have a common left eigenvector* $\boldsymbol{\omega}$ *such that*

$$\boldsymbol{\omega}^{\mathrm{T}} \begin{bmatrix} B_1 & \cdots & B_n \end{bmatrix} = \mathbf{0};$$

and the matrix

$$F_{\mathrm{L}} \triangleq \begin{bmatrix} (A_1 - \lambda_1 I_r) & \cdots & (A_n - \lambda_n I_r) & B_1 & \cdots & B_n \end{bmatrix}$$

is not of full row rank for any eigenvalues $\lambda_1, \ldots, \lambda_n$ *of the state matrix* $A_1, \ldots, A_n$*, respectively. When* $n = 1$*, the n-D F-M model reduces to the conventional 1-D state-space model, and the reduction conditions of Theorems 1 and 2 become that the state matrix* A_1 *has a left eigenvector* $\boldsymbol{\omega}$ *such that*

$$\boldsymbol{\omega}^{\mathrm{T}} B_1 = \mathbf{0},$$

or equivalently the matrix

$$F_{\mathrm{L}} \triangleq \begin{bmatrix} (A_1 - \lambda_1 I_r) & B_1 \end{bmatrix}$$

is not of full row rank for any eigenvalue λ_1 *of the state matrix* A_1*. Satisfying these reduction conditions means that the corresponding 1-D state-space model* $(A_1, B_1, C_1, D; r)$ *is not controllable [38, 60].*

However, when $n \geq 2$*, the situation becomes much more complicated and such a relationship no longer holds. A controllable n-D F-M model may still be reducible by Theorems 5.3 and 5.4, while an irreducible n-D F-M model may even be uncontrollable. To see this more clearly, review one of the most commonly used causal controllability notion [79]: A 2-D F-M model* $(\boldsymbol{A}, \boldsymbol{B}, C, D; r)$ *with* $\boldsymbol{A} = (A_1, A_2)$ *and* $\boldsymbol{B} = (B_1, B_2)$ *is locally controllable if and only if*

$$\begin{bmatrix} I_r - A_1 z_1 - A_2 z_2 & B_1 z_1 + B_2 z_2 \end{bmatrix}$$

is of full rank for any (z_1, z_2) *in* $\mathbf{C} \times \mathbf{C}$*. Consider a controllable 2-D F-M model:*

$$A_1 = A_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad B_1 = B_2 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 \end{bmatrix}, D = 0, \tag{5.43}$$

as

$$\begin{bmatrix} I_r - A_1 z_1 - A_2 z_2 & B_1 z_1 + B_2 z_2 \end{bmatrix} = \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 - z_1 - z_2 & 2z_1 + 2z_2 \end{array}\right]$$

is always of full rank for any (z_1, z_2) *in* $\mathbf{C} \times \mathbf{C}$*. However, this model can be reduced by Theorem 5.2, as for the eigenvalues* $\lambda_1 = \lambda_2 = 0$ *of* A_1 *and* A_2*, the matrix*

$$F_{\mathrm{L}} = \begin{bmatrix} A_1 - \lambda_1 I_2 & A_2 - \lambda_2 I_2 & B_1 & B_2 \end{bmatrix} = \left[\begin{array}{cc|cc|c|c} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 2 & 2 \end{array}\right]$$

is not of full row rank.

On the other hand, one can verify that the 2-D F-M model (3.35) *cannot be reduced, whereas it is uncontrollable as*

$$\begin{aligned}&\left[\begin{array}{cc} I_r - A_1 z_1 - A_2 z_2 & B_1 z_1 + B_2 z_2 \end{array}\right] \\ =&\left[\begin{array}{cc} 1 - 3z_1 - z_2 & z_1 - z_2 \end{array}\right]\end{aligned}$$

is not of full rank for $z_1 = z_2 = \frac{1}{4}$. *Similar discussion can also be carried out for the relationship between the reduction conditions of Theorems 5.1, 5.2 and the observability. Consequently, for n-D* ($n \geq 2$) *F-M models, establishing an explicit relationship between reducibility and controllability/observability is still an open problem.*

Remark 5.7. *It should be noted that once an n-D F-M model with a lowest possible order of a given n-D filter or system is obtained by the proposed exact order reduction approach, it is straightforward to get a corresponding circuit implementation by the well-known techniques (see, e.g., [51, 76]) and the references therein, remark 5 of [78], and thus such details are omitted.*

Remark 5.8. *It should be noted that once an n-D F-M model with a lowest possible order of a given n-D filter or system is obtained by the proposed exact order reduction approach, it is straightforward to get a corresponding circuit implementation by the well-known techniques (see, e.g., [51, 76]) and the references therein, remark 5 of [78], and thus such details are omitted.*

5.1.3 Application Examples

Example 5.4. *Consider the* 4*-D F-M model* $(\boldsymbol{A}, \boldsymbol{B}, C, D; r)$ *given in [16] of*

$$\begin{aligned}&H(z_1, z_2, z_3, z_4) \\ =&\left[\begin{array}{cc} \dfrac{n_{11} z_2 + n_{12} z_3 z_4}{d_{11} z_2 + d_{12} z_3 + d_{13} z_1 z_2 + 1} & \dfrac{n_{21} z_3 + n_{22} z_4}{d_{11} z_2 + d_{12} z_3 + d_{13} z_1 z_2 + 1} \\ \dfrac{z_2 \left(n_{31} z_1 + n_{32} z_3\right)}{d_{21} z_1 + d_{22} z_2 z_4 + d_{23} z_1 z_2 z_3 + 1} & \dfrac{n_{41} z_1 z_2 z_3}{d_{21} z_1 + d_{22} z_2 z_4 + d_{23} z_1 z_2 z_3 + 1} \end{array}\right] \quad (5.44)\end{aligned}$$

as follows:

$$A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -d_{13} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -d_{21} & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} -d_{11} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} n_{11} & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ n_{31} & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} -d_{12} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -d_{23} & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 0 & n_{21} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & n_{41} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ n_{32} & 0 \end{bmatrix},$$

$$A_4 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -d_{22} & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \quad B_4 = \begin{bmatrix} 0 & n_{22} \\ 0 & 0 \\ n_{12} & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad D = \mathbf{0}, \quad r = 13.$$

For this model, the distinct eigenvalues of A_1*,* A_2*,* A_3 *and* A_4 *are* $\{0, -d_{21}\}$*,* $\{0, -d_{11}\}$*,* $\{0, -d_{12}\}$ *and* $\{0\}$*, respectively. It can be checked that the matrix*

$$F_L = \begin{bmatrix} (A_1 - \lambda_1 I_r) & \cdots & (A_4 - \lambda_4 I_r) & B_1 & \cdots & B_4 \end{bmatrix}$$

is not of full row rank for $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0$*, which implies that the given model can be reduced. By the proposed approach, we can finally obtain a new lower-order 4-D F-M model as*

$$\hat{A}_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -d_{13} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -d_{21} & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \hat{B}_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$\hat{A}_2 = \begin{bmatrix} -d_{11} & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \hat{B}_2 = \begin{bmatrix} n_{11} & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ n_{31} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$\hat{A}_3 = \begin{bmatrix} -d_{12} & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -d_{23} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \hat{B}_3 = \begin{bmatrix} 0 & n_{21} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & n_{41} \\ n_{32} & 0 \end{bmatrix},$$

$$\hat{A}_4 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -d_{22} & 0 & 0 & 0 \end{bmatrix}, \quad \hat{B}_4 = \begin{bmatrix} 0 & n_{22} \\ 0 & 0 \\ n_{12} & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$\hat{C} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}, \quad \hat{B}_1 = \mathbf{0}, \quad \hat{D} = \mathbf{0}, \quad \hat{r} = 7.$$

That is, the order of the given 4-D F-M can be reduced by one-half.

Example 5.5. *In this example, we shall apply the proposed common eigenvector approach to reduce a 2-D F-M model for the mental rolling process [57, 80] shown in Figure 5.1. Such a process can be described by the equation [57, 80]:*

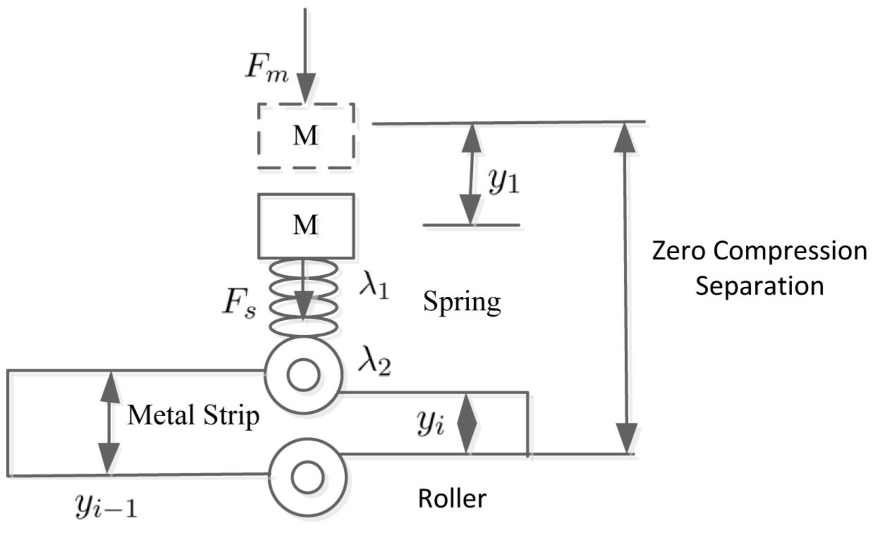

Figure 5.1: Mental rolling process.

$$y_i(t) = \frac{\lambda}{\lambda + Mp^2} \left\{ \left(1 + \frac{Mp^2}{\lambda_1}\right) y_{i-1}(t) - \frac{1}{\lambda_2} F_m \right\},$$

where p denotes the differentiation operator $d/d(t)$; $y_i(t)$ is the ith actual roll-gap thickness; F_m is the force developed by the motor and M is the lumped mass of the roll-gap adjusting mechanism; λ_1 is the stiffness of the adjusting mechanism spring; λ_2 is the hardness of the mental strip and $\lambda = \frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2}$ is the composite stiffness of the mental strip and the roll mechanism.

As shown in [57], the mental rolling can be described by the following 2-D F-M model:

$$A_1 = \begin{bmatrix} a_3 & a_4 & a_1 & a_2 & a_5 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \; A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ a_3 & a_4 & a_1 & a_2 & a_5 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}, \; B_1 = \begin{bmatrix} b \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \; B_2 = \begin{bmatrix} 0 \\ 0 \\ b \\ 0 \\ 0 \end{bmatrix}, \tag{5.45}$$

$$C = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \quad D = 0, \quad r = 5$$

with

$$a_1 = \frac{2M}{\lambda T_1^2 + M}, \qquad a_2 = \frac{-M}{\lambda T_1^2 + M},$$

$$a_3 = \frac{\lambda}{\lambda T_1^2 + M}\left(T_1^2 + \frac{M}{\lambda}\right), \qquad a_4 = \frac{-2\lambda M}{\lambda_1\left(\lambda T_1^2 + M\right)},$$

$$a_5 = \frac{\lambda M}{\lambda_1\left(\lambda T_1^2 + M\right)}, \qquad b = \frac{-\lambda T_1^2}{\lambda_2\left(\lambda T_1^2 + M\right)}.$$

For this 2-D F-M model, the distinct eigenvalues of A_1 and A_2 are $\{0, a_3\}$ and $\{0, \frac{a_1 \pm \sqrt{a_1^2+4a_2}}{2}\}$, respectively. One can check that

$$F_R = \begin{bmatrix} (A_1 - \lambda_1 I_5) \\ (A_2 - \lambda_2 I_5) \\ C \end{bmatrix}$$

is not of full column rank for $\lambda_1 = \lambda_2 = 0$, which means that the 2-D F-M model can be reduced. Applying the proposed approach to (5.45) yields a new lower-order F-M model:

$$\hat{A}_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ \frac{a_3a_4}{a_5} & \frac{a_1a_3}{a_5} & a_3 \end{bmatrix}, \quad \hat{A}_2 = \begin{bmatrix} 0 & 0 & 0 \\ a_4 & a_1 & a_5 \\ 1 & \frac{a_2}{a_5} & 0 \end{bmatrix}, \quad \hat{B}_1 = \begin{bmatrix} 0 \\ 0 \\ \frac{a_3b}{a_5} \end{bmatrix}, \quad \hat{B}_2 = \begin{bmatrix} 0 \\ b \\ 0 \end{bmatrix}, \tag{5.46}$$

$$\hat{C} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}, \quad \hat{D} = 0, \quad \hat{r} = 3.$$

It is seen that the order of the obtained F-M model of (5.46) *is only three-fifths of the given model of* (5.45).

Example 5.6. *Consider the following 3-D F-M model for implementations in distributed grid sensor networks [8]:*

$$A_1 = \begin{bmatrix} 0 & 0 & -d_{101} & -d_{011} & 0 & -d_{202} & 0 & -d_{112} & 0 & -d_{303} & 0 & -d_{213} & 0 & -d_{314} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -d_{101} & -d_{011} & 0 & -d_{202} & 0 & -d_{112} & 0 & -d_{303} & 0 & -d_{213} & 0 & -d_{314} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&1&0 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0 \end{bmatrix},$$

$$C = \begin{bmatrix} n_{100} & 0 & 0 & 0 & n_{201} & 0 & n_{111} & 0 & 0 & 0 & n_{212} & 0 & 0 & 0 \end{bmatrix}, \; D = 0,$$

Applying the proposed constructive reduction procedure yields the following much lower F-M model:

$$\hat{A}_1 = \begin{bmatrix} 0 & -d_{101} & 0 & -d_{202} & 0 & -d_{303} \\ 0&0&0&0&0&0 \\ 0&1&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&1&0&0 \\ 0&0&0&0&0&0 \end{bmatrix}, \quad \hat{A}_2 = \begin{bmatrix} 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \end{bmatrix}$$

$$\hat{A}_3 = \begin{bmatrix} 0&0&0&0&0&0 \\ 1&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&1&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&1&0 \end{bmatrix}, \; \hat{B}_1 = \begin{bmatrix} 1\\0\\0\\0\\0\\0 \end{bmatrix}, \; \hat{B}_2 = \begin{bmatrix} 0\\0\\0\\0\\0\\0 \end{bmatrix}, \; \hat{B}_3 = \begin{bmatrix} 0\\0\\0\\0\\0\\0 \end{bmatrix},$$

$$\hat{C} = \begin{bmatrix} n_{100} & 0 & n_{201} & 0 & 0 & 0 \end{bmatrix}, \; \hat{D} = 0$$

5.2 Reduction of Roesser Models with Constrained Common Eigenvectors

In this section, new reducibility conditions and the corresponding reduction procedure will be developed for the n-D Roesser model by taking into account the eigenvalues of all the main diagonal blocks in the system matrix. To do this, we first introduce the notion of the constrained common eigenvector and give some notational preparations.

For an n-D Roesser model with $\boldsymbol{r} = (r_1, \ldots, r_n)$ and $r = r_1 + \ldots + r_n$, the set

$$\{\boldsymbol{e}_{1,1},\ \ldots,\ \boldsymbol{e}_{1,r_1},\ \boldsymbol{e}_{2,1},\ \ldots,\ \boldsymbol{e}_{2,r_2},\ \ldots,\ \boldsymbol{e}_{n,1},\ \ldots,\ \boldsymbol{e}_{n,r_n}\} \tag{5.47}$$

is called the associated standard basis of this model, where the components of $\boldsymbol{e}_{i,k}$, $i = 1, \ldots, n$, $k = 1, \ldots, r_i$, are 0 except for the $s_{i,k}$th component being 1 with

$$s_{i,k} = \begin{cases} k, & i = 1; \\ k + \sum_{t=1}^{i-1} r_t, & i > 1. \end{cases}$$

Then, for $i = 1, \ldots, n$, let

$$E_i = \begin{bmatrix} \mathbf{0} & \ldots & \mathbf{0} & \boldsymbol{e}_{i,1} & \ldots & \boldsymbol{e}_{i,r_i} & \mathbf{0} & \ldots & \mathbf{0} \end{bmatrix}, \tag{5.48}$$

whose columns are zero except the $s_{i,k}$th columns being $\boldsymbol{e}_{i,k}$, $k = 1, \ldots, r_i$.

For the given n-D Roesser model $(A, B, C, D; \boldsymbol{r})$, we also define

$$A_{\mathrm{c}i} = AE_i = \begin{bmatrix} 0 & \ldots & A_{1,i} & \ldots & 0 \\ 0 & \ldots & A_{2,i} & \ldots & 0 \\ & & \vdots & & \\ 0 & \ldots & \vdots & \ldots & 0 \\ 0 & \ldots & A_{n,i} & \ldots & 0 \end{bmatrix}, \quad C_{\mathrm{c}i} = CE_i = \begin{bmatrix} 0 & \ldots & C_i & \ldots & 0 \end{bmatrix}, \tag{5.49a}$$

$$A_{\mathrm{r}i} = E_i^{\mathrm{T}} A = \begin{bmatrix} 0 & 0 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ A_{i,1} & A_{i,2} & \ldots & A_{i,n} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & 0 \end{bmatrix}, \quad B_{\mathrm{r}i} = E_i^{\mathrm{T}} B = \begin{bmatrix} 0 \\ \vdots \\ B_i \\ \vdots \\ 0 \end{bmatrix}, \tag{5.49b}$$

for $i = 1, \ldots, n$, where E_i is defined in (5.48) and the subscripts c and r denote that the coefficient matrices are partitioned based on columns and rows, respectively.

5.2.1 Reduction Using Constrained Common Right Eigenvectors

In this subsection, reducibility conditions and the corresponding reduction procedure are given by employing the constrained common right eigenvectors.

Theorem 5.5. *For a given n-D Roesser model $(A, B, C, D; \boldsymbol{r})$, if the matrices $A_{\mathrm{c}1}, \ldots, A_{\mathrm{c}n}$ have a common right eigenvector constrained by the matrices $C_{\mathrm{c}1}, \ldots, C_{\mathrm{c}n}$ with $A_{\mathrm{c}i}$ and $C_{\mathrm{c}i}$ defined in* (5.49a), *$i \in \{1, \ldots, n\}$, then $(A, B, C, D; \boldsymbol{r})$ is reducible.*

Proof. Suppose that $A_{\mathrm{c}1}, \ldots, A_{\mathrm{c}n}$ have a common right eigenvector $\boldsymbol{\omega}$ constrained by $C_{\mathrm{c}1}, \ldots, C_{\mathrm{c}n}$. Then, for every $i \in \{1, \ldots, n\}$ there exists an eigenvalue λ_i of $A_{\mathrm{c}i}$ such that

$$A_{\mathrm{c}i}\,\boldsymbol{\omega} = \lambda_i\,\boldsymbol{\omega}, \quad C_{\mathrm{c}i}\,\boldsymbol{\omega} = \mathbf{0}. \tag{5.50}$$

We can express this eigenvalue λ_i and the corresponding eigenvector as

$$\lambda_i = \alpha_i + j\beta_i,\ \boldsymbol{\omega} = \boldsymbol{\mu} + j\boldsymbol{\nu}, \tag{5.51}$$

where j denotes the imaginary unit, α_i and β_i are the real and imaginary parts of λ_i, and $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ the real and imaginary parts of $\boldsymbol{\omega}$, respectively.

Note that when the eigenvector $\boldsymbol{\omega}$ corresponds to a real eigenvalue λ_i, $i \in \{1,\ldots,n\}$, we have that β_i and $\boldsymbol{\nu}$ are both zero, i.e., $\lambda_i = \alpha_i, \quad \boldsymbol{\omega} = \boldsymbol{\mu}$. It follows from (5.50) and (5.51) that

$$A_{\mathrm{c}i}\boldsymbol{\mu} = \alpha_i\boldsymbol{\mu} - \beta_i\boldsymbol{\nu}, \quad A_{\mathrm{c}i}\boldsymbol{\nu} = \alpha_i\boldsymbol{\nu} + \beta_i\boldsymbol{\mu}, \tag{5.52a}$$

$$C_{\mathrm{c}i}\boldsymbol{\mu} = \mathbf{0},\ \ C_{\mathrm{c}i}\boldsymbol{\nu} = \mathbf{0}. \tag{5.52b}$$

Construct a nonsingular matrix $T \in \mathbf{R}^{r\times r}$ as

$$T = \begin{bmatrix}\boldsymbol{\mu} & \boldsymbol{\nu} \,\big|\, \hat{\boldsymbol{e}}_{1,1} \ \ldots\ \hat{\boldsymbol{e}}_{1,\hat{r}_1}\ \ldots\ \hat{\boldsymbol{e}}_{n,1}\ \ldots\ \hat{\boldsymbol{e}}_{n,\hat{r}_n}\end{bmatrix} \triangleq \begin{bmatrix}\tilde{R}\,\big|\,R\end{bmatrix}, \tag{5.53}$$

without including $\boldsymbol{\nu}$ if $\boldsymbol{\nu} = \mathbf{0}$, where every $\hat{\boldsymbol{e}}_{i,k}$ is selected from $\{\boldsymbol{e}_{i,1},\ldots,\boldsymbol{e}_{i,r_i}\}$ in (5.47) with $i \in \{1,\ldots,n\}$ and $k \in \{1,\ldots,\hat{r}_i\}$; $\tilde{R} \in \mathbf{R}^{r\times\tilde{r}}$, $R \in \mathbf{R}^{r\times\hat{r}}$ with $\hat{r} = \hat{r}_1 + \ldots + \hat{r}_n$ and $\tilde{r} = r - \hat{r}$. Then, partition T^{-1} as

$$T^{-1} \triangleq \begin{bmatrix}\tilde{L}\\ L\end{bmatrix} \tag{5.54}$$

with $\tilde{L} \in \mathbf{R}^{\tilde{r}\times r}$ and $L \in \mathbf{R}^{\hat{r}\times r}$.

In view of (5.52a), we have that for every $i \in \{1,\ldots,n\}$, $A_{\mathrm{c}i}\boldsymbol{\mu}$ and $A_{\mathrm{c}i}\boldsymbol{\nu}$ can be expressed as a linear combination of the vectors $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$, which gives that

$$\begin{aligned} A_{\mathrm{c}i}T &= \begin{bmatrix}A_{\mathrm{c}i}\,\boldsymbol{\mu}\ A_{\mathrm{c}i}\boldsymbol{\nu}\ A_{\mathrm{c}i}\hat{\boldsymbol{e}}_{1,1}\ \ldots\ A_{\mathrm{c}i}\hat{\boldsymbol{e}}_{1,\hat{r}_1}\ \ldots\ A_{\mathrm{c}i}\hat{\boldsymbol{e}}_{n,1}\ \ldots\ A_{\mathrm{c}i}\hat{\boldsymbol{e}}_{n,\hat{r}_n}\end{bmatrix}\\ &= \begin{bmatrix}\boldsymbol{\mu}\ \boldsymbol{\nu}\ \hat{\boldsymbol{e}}_{1,1}\ \ldots\ \hat{\boldsymbol{e}}_{1,\hat{r}_1}\ \ldots\ \hat{\boldsymbol{e}}_{n,1}\ \ldots\ \hat{\boldsymbol{e}}_{n,\hat{r}_n}\end{bmatrix}\begin{bmatrix}\tilde{A}_{\mathrm{c}i} & \check{A}_{\mathrm{c}i}\\ \mathbf{0} & \hat{A}_{\mathrm{c}i}\end{bmatrix} = T\begin{bmatrix}\tilde{A}_{\mathrm{c}i} & \check{A}_{\mathrm{c}i}\\ \mathbf{0} & \hat{A}_{\mathrm{c}i}\end{bmatrix}, \end{aligned} \tag{5.55}$$

for some $\tilde{r}\times\tilde{r}$ matrix $\tilde{A}_{ci}$ and

$$\check{A}_{\mathrm{c}i} = \tilde{L}A_{\mathrm{c}i}R \in \mathbf{R}^{\tilde{r}\times\hat{r}},\ \hat{A}_{ci} = LA_{\mathrm{c}i}R \in \mathbf{R}^{\hat{r}\times\hat{r}} \tag{5.56}$$

with $\hat{r} = \hat{r}_1 + \ldots + \hat{r}_n < r$. By the definitions of $\boldsymbol{e}_{i,k}$ and R, we have

$$ZR = \left[\hat{\boldsymbol{e}}_{1,1}z_1 \ldots \hat{\boldsymbol{e}}_{1,\hat{r}_1}z_1 \ldots \hat{\boldsymbol{e}}_{n,1}z_n \ldots \hat{\boldsymbol{e}}_{n,\hat{r}_n}z_n\right] = R\hat{Z}, \tag{5.57}$$

with $\hat{Z} = \mathrm{diag}\{z_1 I_{\hat{r}_1}, \ldots, z_n I_{\hat{r}_n}\}$. Thus,

$$\begin{aligned} T^{-1}AZT &= T^{-1}\left(\sum_{i=1}^{n} A_{\mathrm{c}i}z_i\right)T = \sum_{i=1}^{n} T^{-1}A_{\mathrm{c}i}Tz_i \\ &= \begin{bmatrix} \sum_{i=1}^{n}\tilde{A}_{\mathrm{c}i}z_i & \sum_{i=1}^{n}\check{A}_{\mathrm{c}i}z_i \\ \mathbf{0} & \sum_{i=1}^{n}\hat{A}_{\mathrm{c}i}z_i \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{n}\tilde{A}_{\mathrm{c}i}z_i & \tilde{L}AZR \\ \mathbf{0} & LAZR \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{n}\tilde{A}_{\mathrm{c}i}z_i & \tilde{L}AR\hat{Z} \\ \mathbf{0} & LAR\hat{Z} \end{bmatrix}. \end{aligned} \tag{5.58}$$

It follows from (5.52b) that for every $i \in \{1, \ldots, n\}$,

$$\begin{aligned} C_{\mathrm{c}i}T = \big[\ & C_{\mathrm{c}i}\boldsymbol{\mu} \ \ C_{\mathrm{c}i}\boldsymbol{\nu} \ \ C_{\mathrm{c}i}\hat{\boldsymbol{e}}_{1,1} \ \ \ldots \ \ C_{\mathrm{c}i}\hat{\boldsymbol{e}}_{1,\hat{r}_1} \ \ \ldots \\ & C_{\mathrm{c}i}\hat{\boldsymbol{e}}_{n,1} \ \ \ldots \ \ C_{\mathrm{c}i}\hat{\boldsymbol{e}}_{n,\hat{r}_n} \ \big] = \big[\ \mathbf{0} \ \ \hat{C}_{\mathrm{c}i}\ \big], \end{aligned} \tag{5.59}$$

with $\hat{C}_{\mathrm{c}i} = C_{\mathrm{c}i}R \in \mathbf{R}^{p\times\hat{r}}$. Thus,

$$CZT = \left(\sum_{i=1}^{n} C_{\mathrm{c}i}z_i\right)T = \left[\mathbf{0} \ \ \sum_{i=1}^{n}C_{\mathrm{c}i}z_i\right] = \left[\mathbf{0} \ CZR\right] = \left[\mathbf{0} \ CR\hat{Z}\right]. \tag{5.60}$$

We see from (5.58) and (5.60) that

$$\begin{aligned} CZ(I_r - AZ)^{-1}B &= CZTT^{-1}(I_r - AZ)^{-1}TT^{-1}B \\ &= (CZT)\left(\ I_r - T^{-1}AZT\ \right)^{-1}(T^{-1}B) \\ &= \left[\mathbf{0} \ CR\hat{Z}\right]\begin{bmatrix} I_2 - \sum_{i=1}^{n}\tilde{A}_{\mathrm{c}i}z_i & -\tilde{L}AR\hat{Z} \\ \mathbf{0} & I_{\hat{r}} - LAR\hat{Z} \end{bmatrix}^{-1}\begin{bmatrix} \tilde{L}B \\ LB \end{bmatrix} \\ &= CR\hat{Z}\left(\ I_{\hat{r}} - LAR\hat{Z}\ \right)^{-1}LB. \end{aligned} \tag{5.61}$$

That is to say, we have obtained a new n-D Roesser model

$$(\hat{A}, \hat{B}, \hat{C}, D, \hat{\boldsymbol{r}}) \triangleq (LAR, LB, CR, D; \hat{\boldsymbol{r}})$$

with $\hat{r} < r$. □

The reduction condition based on common right eigenvector given in Theorem 5.5 can be equivalently transformed into the following one characterized by eigenvalues of $A_{\mathrm{c}1}, \ldots, A_{\mathrm{c}n}$.

Theorem 5.6. *For a given n-D Roesser model* $(A, B, C, D; \boldsymbol{r})$*, if the matrix*

$$F_{\mathrm{c}} \triangleq \begin{bmatrix} (A_{\mathrm{c}1} - \lambda_1 I_r) \\ \vdots \\ (A_{\mathrm{c}n} - \lambda_n I_r) \\ C_{\mathrm{c}1} \\ \vdots \\ C_{\mathrm{c}n} \end{bmatrix} \tag{5.62}$$

is not of full column rank for some eigenvalue λ_i *of* $A_{\mathrm{c}i}$ *with* $A_{\mathrm{c}i}$ *and* $C_{\mathrm{c}i}$ *defined in* (5.49a)*,* $i \in \{1, \ldots, n\}$*, then* $(A, B, C, D; \boldsymbol{r})$ *is reducible.*

Proof. By the definition of constrained common right eigenvector, we have that matrices $A_{\mathrm{c}1}$, …, $A_{\mathrm{c}n}$ have a common right eigenvector $\boldsymbol{\omega}$ constrained by $C_{\mathrm{c}1}, \ldots, C_{\mathrm{c}n}$ if and only if there exists an eigenvalue λ_i of $A_{\mathrm{c}i}$, $i \in \{1, \ldots, n\}$, such that

$$A_{\mathrm{c}i}\boldsymbol{\omega} = \lambda_i \boldsymbol{\omega},\ C_{\mathrm{c}i}\boldsymbol{\omega} = \mathbf{0},\ i = 1, \ldots, n. \tag{5.63}$$

(5.63) is equivalent to $F_{\mathrm{c}}\boldsymbol{\omega} = \mathbf{0}$, which is then equivalent to the rank deficient of the matrix F_{c} in (5.62). Thus, we have that $A_{\mathrm{c}1}$, …, $A_{\mathrm{c}n}$ have a common right eigenvector $\boldsymbol{\omega}$ constrained by $C_{\mathrm{c}1}, \ldots, C_{\mathrm{c}n}$ if and only if the matrix F_{c} in (5.62) is not of full column rank for some eigenvalue λ_i of $A_{\mathrm{c}i}$, $i = 1, \ldots, n$. In view of Theorem 5.5, we obtain Theorem 5.6. □

Remark 5.9. *In view of the definition* $A_{\mathrm{c}i}$ *in* (5.49a)*, one can verify that the eigenvalues of* $A_{\mathrm{c}i}$ *include all the eigenvalues of the diagonal sub-block* $A_{i,i}$ *in the system matrix* A*. It means that the eigenvalues of all the blocks* $A_{i,i}$*,* $i = 1, \ldots, n$*, have been taken into account in Theorem 5.6.*

Remark 5.10. *The proof of Theorem 5.6 indicates a method to obtain a common right eigenvector* $\boldsymbol{\omega}$ *of* $A_{\mathrm{c}1}, \ldots, A_{\mathrm{c}n}$ *constrained by* $C_{\mathrm{c}1}, \ldots, C_{\mathrm{c}n}$*. First, select an eigenvalue* λ_i *of* $A_{\mathrm{c}i}$*,* $i = 1, \ldots, n$*, such that the matrix* F_{c} *in (5.62) is not of full column rank. Second, find a nonzero vector* $\boldsymbol{\omega}$ *such that* $F_{\mathrm{c}}\boldsymbol{\omega} = \mathbf{0}$*. However, in this way, we need to find all the eigenvalues of* $A_{\mathrm{c}1}, \ldots, A_{\mathrm{c}n}$*, and consider all the possible combination of these eigenvalues. It is therefore desirable to have a method to directly compute the constrained common eigenvector without prior knowledge of these eigenvalues, which will be discussed in the next section.*

Now, a basic reduction procedure can be given as follows.

Procedure 5.2: Exact Order Reduction of an n-D Roesser Model Using a Constrained Common Right Eigenvector

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Input : A given n-D Roesser model $(A, B, C, D; \boldsymbol{r})$;
Output: A reduced-order n-D Roesser model $(\hat{A}, \hat{B}, \hat{C}, D; \hat{\boldsymbol{r}})$;

while $A_{c1}, \ldots, A_{cn}$ *share a common right eigenvector* $\boldsymbol{\omega}$ *constrained by* $C_{c1}, \ldots, C_{cn}$ **do**

  Step 1: Express $\boldsymbol{\omega}$ as $\boldsymbol{\omega} = \boldsymbol{\mu} + j\boldsymbol{\nu}$ and select $\hat{\boldsymbol{e}}_{i,k}$ from $\{\boldsymbol{e}_{i,1}, \ldots, \boldsymbol{e}_{i,r_i}\}$ in (5.47) to construct a nonsingular matrix T in the form of (5.53);

  Step 2: Extract R and L from T of (5.53) and T^{-1} of (5.54), respectivley;

  Step 3: Obtain a new Roesser model $(\hat{A}, \hat{B}, \hat{C}, D; \hat{\boldsymbol{r}})$:

$$\hat{A} \triangleq LAR, \quad \hat{B} \triangleq LB, \quad \hat{C} \triangleq CR; \tag{5.64}$$

  Renew $(A, B, C, D; \boldsymbol{r}) \triangleq (\hat{A}, \hat{B}, \hat{C}, D; \hat{\boldsymbol{r}})$;

end

return $(\hat{A}, \hat{B}, \hat{C}, D; \hat{\boldsymbol{r}}) \triangleq (A, B, C, D; \boldsymbol{r})$,

---

Example 5.7. *To show the details and effectiveness of the proposed procedure, consider the 3-D Roesser model:*

$$A = \left[\begin{array}{cccc|cccc|cc}
-1 & 1 & -1 & 0 & 3 & 2 & 0 & 0 & 1 & 0 \\
-2 & 1 & -3 & 0 & 1 & 4 & 0 & -1 & 1 & 1 \\
-2 & 0 & 4 & 1 & 1 & 0 & 1 & 0 & 2 & 0 \\
-4 & 2 & -4 & 1 & 7 & 8 & 1 & 4 & 2 & 0 \\
\hline
-1 & 1 & 0 & 0 & 3 & 2 & 0 & 0 & 1 & -1 \\
-1 & 0 & 0 & 0 & -2 & 3 & 0 & 0 & 0 & 0 \\
-3 & -1 & 0 & 2 & 1 & -2 & 4 & 0 & -3 & -1 \\
1 & 0 & 1 & 0 & 1 & -1 & -1 & 2 & 2 & 1 \\
\hline
-2 & 0 & 1 & 1 & 1 & 2 & 1 & 2 & 1 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right], \quad B = \left[\begin{array}{c} 1 \\ 1 \\ 1 \\ 1 \\ \hline -1 \\ 0 \\ -1 \\ 1 \\ \hline 0 \\ 1 \end{array}\right], \tag{5.65}$$

$$C = \left[\begin{array}{cccc|cccc|cc} -2 & 0 & 0 & 1 & 2 & 0 & 2 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \end{array}\right], \quad D = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \boldsymbol{r} = (4, 4, 2).$$

For this model, we have the matrices

$$A_{c1} = \begin{bmatrix} A_{1,1} & \mathbf{0}_{4,4} & \mathbf{0}_{4,2} \\ A_{2,1} & \mathbf{0}_{4,4} & \mathbf{0}_{4,2} \\ A_{3,1} & \mathbf{0}_{2,4} & \mathbf{0}_{2,2} \end{bmatrix}, \quad A_{c2} = \begin{bmatrix} \mathbf{0}_{4,4} & A_{1,2} & \mathbf{0}_{4,2} \\ \mathbf{0}_{4,4} & A_{2,2} & \mathbf{0}_{4,2} \\ \mathbf{0}_{2,4} & A_{3,2} & \mathbf{0}_{2,2} \end{bmatrix}, \quad A_{c3} = \begin{bmatrix} \mathbf{0}_{4,4} & \mathbf{0}_{4,4} & A_{1,3} \\ \mathbf{0}_{4,4} & \mathbf{0}_{4,4} & A_{2,3} \\ \mathbf{0}_{2,4} & \mathbf{0}_{2,4} & A_{3,3} \end{bmatrix},$$
$$C_{c1} = \begin{bmatrix} C_1 & \mathbf{0}_{2,4} & \mathbf{0}_{2,2} \end{bmatrix}, \quad C_{c2} = \begin{bmatrix} \mathbf{0}_{2,4} & C_2 & \mathbf{0}_{2,2} \end{bmatrix}, \quad C_{c3} = \begin{bmatrix} \mathbf{0}_{2,4} & \mathbf{0}_{2,4} & C_3 \end{bmatrix}. \tag{5.66}$$

It can be verified that corresponding to the eigenvalues $\lambda_1 = -j$, $\lambda_2 = 3 - 2j$ *and* $\lambda_3 = 0$

of A_{c1}, A_{c2} and A_{c3}, respectively, the vector

$$\boldsymbol{\omega} = \begin{bmatrix} j & -1+j & 0 & 2j & j & -1 & -j & 1 & 0 & 0 \end{bmatrix}^{\mathrm{H}} \tag{5.67}$$

is a common right eigenvector of A_{c1}, A_{c2}, A_{c3} constrained by C_{c1}, C_{c2}, C_{c3}. Thus, this model can be reduced by applying the proposed reduction procedure.

Step 1: *Since the constrained common right eigenvector $\boldsymbol{\omega}$ is complex, we have $\boldsymbol{\omega} = \boldsymbol{\mu} + j\boldsymbol{\nu}$ with*

$$\boldsymbol{\mu} = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \end{bmatrix}^{\mathrm{T}}, \tag{5.68a}$$

$$\boldsymbol{\nu} = \begin{bmatrix} -1 & -1 & 0 & -2 & -1 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}^{\mathrm{T}}. \tag{5.68b}$$

Then, we can construct a nonsingular matrix

$$T = \left[\begin{array}{cc|cccc:cc:cc} \boldsymbol{\mu} & \boldsymbol{\nu} & \hat{\boldsymbol{e}}_{1,1} & \hat{\boldsymbol{e}}_{1,2} & \hat{\boldsymbol{e}}_{1,3} & \hat{\boldsymbol{e}}_{1,4} & \hat{\boldsymbol{e}}_{2,1} & \hat{\boldsymbol{e}}_{2,2} & \hat{\boldsymbol{e}}_{3,1} & \hat{\boldsymbol{e}}_{3,2} \end{array}\right] = \left[\tilde{R} \middle| R\right], \tag{5.69}$$

where $\hat{\boldsymbol{e}}_{1,1} = \boldsymbol{e}_{1,1}$, $\hat{\boldsymbol{e}}_{1,2} = \boldsymbol{e}_{1,2}$, $\hat{\boldsymbol{e}}_{1,3} = \boldsymbol{e}_{1,3}$, $\hat{\boldsymbol{e}}_{1,4} = \boldsymbol{e}_{1,4}$, $\hat{\boldsymbol{e}}_{2,1} = \boldsymbol{e}_{2,1}$, $\hat{\boldsymbol{e}}_{2,2} = \boldsymbol{e}_{2,2}$, $\hat{\boldsymbol{e}}_{3,1} = \boldsymbol{e}_{3,1}$, $\hat{\boldsymbol{e}}_{3,2} = \boldsymbol{e}_{3,2}$.

Step 2: *Extract L from T^{-1}:*

$$T^{-1} = \left[\begin{array}{cccccccccc} 0&0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&1&0&0&0 \\ \hline 1&0&0&0&0&0&1&0&0&0 \\ 0&1&0&0&0&0&1&1&0&0 \\ 0&0&1&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&2&0&0&0 \\ 0&0&0&0&1&0&1&0&0&0 \\ 0&0&0&0&0&1&0&1&0&0 \\ 0&0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&0&0&1 \end{array}\right] \triangleq \begin{bmatrix} \tilde{L} \\ L \end{bmatrix}. \tag{5.70}$$

Step 3: *By (5.64) a new lower-order 3-D Roesser state-space model $(\hat{A}, \hat{B}, \hat{C}, D; \hat{\boldsymbol{r}})$ is obtained as*

$$\hat{A} = \left[\begin{array}{cccc|cc|cc} -4&0&-1&2&4&0&-2&-1 \\ -4&0&-2&2&3&1&0&1 \\ -2&0&4&1&1&0&2&0 \\ -10&0&-4&5&9&4&-4&-2 \\ \hline -4&0&0&2&4&0&-2&-2 \\ 0&0&1&0&-1&2&2&1 \\ \hline -2&0&1&1&1&2&1&0 \\ 0&0&-1&0&0&0&0&1 \end{array}\right], \quad \hat{B} = \left[\begin{array}{c} 0 \\ 1 \\ 1 \\ -1 \\ \hline -2 \\ 1 \\ \hline 0 \\ 1 \end{array}\right],$$

$$\hat{C} = \left[\begin{array}{cccc|cc|cc} -2&0&0&1&2&0&1&1 \\ 0&0&1&0&0&1&0&0 \end{array}\right], \quad D = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \hat{\boldsymbol{r}} = (4, 2, 2). \tag{5.71}$$

By renewing $(A,B,C,D;\boldsymbol{r})$ as $(A,B,C,D;\boldsymbol{r}) \triangleq (\hat{A},\hat{B},\hat{C},D;\hat{\boldsymbol{r}})$ and applying again the proposed reduction procedure to the Roesser model $(A,B,C,D;\boldsymbol{r})$, we can obtain a further lower 3-D Roesser model:

$$\hat{A} = \left[\begin{array}{cc|cc|cc} 1 & 1 & -\frac{1}{2} & -2 & 0 & 0 \\ -2 & 4 & 1 & 0 & 2 & 0 \\ \hline -4 & 0 & 4 & 0 & -2 & -2 \\ 0 & 1 & -1 & 2 & 2 & 1 \\ \hline -2 & 1 & 1 & 2 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \end{array}\right], \quad \hat{B} = \left[\begin{array}{c} \frac{1}{2} \\ 1 \\ \hline -2 \\ 1 \\ \hline 0 \\ 1 \end{array}\right], \tag{5.72}$$

$$\hat{C} = \left[\begin{array}{cc|cc|cc} -2 & 0 & 2 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{array}\right], \quad D = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \hat{\boldsymbol{r}} = (2,2,2).$$

5.2.2 Reduction Using Constrained Common Left Eigenvectors

In the similar way shown in the previous subsection, reducibility conditions can also be established based on the constrained common left eigenvectors.

Theorem 5.7. *For a given n-D Roesser model $(A,B,C,D;\boldsymbol{r})$, if $A_{\mathrm{r}1},\ldots,A_{\mathrm{r}n}$ have a common left eigenvector constrained by $B_{\mathrm{r}1},\ldots,B_{\mathrm{r}n}$ with $A_{\mathrm{r}i}$ and $B_{\mathrm{r}i}$ defined in* (5.49b)*, $i \in \{1,\ldots,n\}$, then $(A,B,C,D;\boldsymbol{r})$ is reducible.*

Theorem 5.8. *For a given n-D Roesser model $(\boldsymbol{A},\boldsymbol{B},C,D;\boldsymbol{r})$, if the matrix*

$$F_{\mathrm{r}} \triangleq \begin{bmatrix} (A_{\mathrm{r}1}-\lambda_1 I_r) & \ldots & (A_{\mathrm{r}n}-\lambda_n I_r) & B_{\mathrm{r}1} & \ldots & B_{\mathrm{r}n} \end{bmatrix} \tag{5.73}$$

is not of full row rank for some eigenvalue λ_i of $A_{\mathrm{r}i}$, with $A_{\mathrm{r}i}$ and $B_{\mathrm{r}i}$ defined in (5.49b)*, $i = 1,\ldots,n$, then $(\boldsymbol{A},\boldsymbol{B},C,D;\boldsymbol{r})$ is reducible.*

Since Theorems 5.7 and 5.8 can be proved similarly as Theorems 5.5 and 5.6, respectively, the details are omitted here for brevity.

Remark 5.11. *Due to the duality, an n-D Roesser model $(A,B,C,\ D;\boldsymbol{r})$ satisfying the reducibility condition of Theorem 5.7 can be reduced as follows: First, use Procedure 5.2 to reduce the Roesser model $(\tilde{A},\tilde{B},\tilde{C},\tilde{D};\boldsymbol{r}) \triangleq (A^{\mathrm{T}},C^{\mathrm{T}},B^{\mathrm{T}},D^{\mathrm{T}};\boldsymbol{r})$ to get a lower-order Roesser model $(\hat{\tilde{A}},\hat{\tilde{B}},\hat{\tilde{C}},\tilde{D};\hat{\boldsymbol{r}})$. Second, set $(\hat{A},\hat{B},\hat{C},D;\hat{\boldsymbol{r}}) \triangleq (\hat{\tilde{A}}^{\mathrm{T}},\hat{\tilde{C}}^{\mathrm{T}},\hat{\tilde{B}}^{\mathrm{T}},D;\hat{\boldsymbol{r}})$ which is a lower-order n-D Roesser model for the given n-D Roesser model $(A,B,C,D;\boldsymbol{r})$.*

Remark 5.12. *Theorems 5.5, 5.6 and Theorems 5.7, 5.8 can be viewed as a kind of generalization of PBH tests for the exact reducibility of n-D Roesser models.*

It should be noted that Theorems 1, 2 and Theorems 3, 4 give only sufficient reduction conditions for the n-D Roesser models. That is to say, not satisfying these conditions does not mean that the Roesser model under consideration is no longer reducible. In fact, it is interesting to see that, for an n-D Roesser model that does not satisfy the reduction conditions, when it is transformed to another equivalent n-D Roesser model in the sense of the input/output equivalence [15, 23, 29], the transformed system may satisfy the reduction conditions, and thus can be further reduced by our approach. To show this fact, some details are given below.

Example 5.8. *Consider the 2-D Roesser model given by,*

$$A = \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 2 \end{array}\right], \quad B = \left[\begin{array}{cc} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ \hline 0 & 0 \end{array}\right], \quad C = \left[\begin{array}{c} 2 \\ 0 \\ 0 \\ \hline 2 \end{array}\right]^{\mathrm{T}}, \quad D = \begin{bmatrix} 0 & 0 \end{bmatrix}, \tag{5.74}$$

with $\boldsymbol{r} = (3,1)$*, which cannot be reduced by the methods of [30, 31, 34]. For this Roesser model, we have*

$$A_{\mathrm{c}1} = \begin{bmatrix} A_{1,1} & \mathbf{0}_{3,1} \\ A_{2,1} & \mathbf{0}_{1,1} \end{bmatrix}, \quad A_{\mathrm{c}2} = \begin{bmatrix} \mathbf{0}_{3,3} & A_{1,2} \\ \mathbf{0}_{1,3} & A_{2,2} \end{bmatrix}, \quad C_{\mathrm{c}1} = \begin{bmatrix} C_1 & \mathbf{0}_{1,1} \end{bmatrix}, \quad C_{\mathrm{c}1} = \begin{bmatrix} \mathbf{0}_{1,3} & C_2 \end{bmatrix}, \tag{5.75}$$

and the distinct eigenvalues of $A_{\mathrm{c}1}$ *and* $A_{\mathrm{c}2}$ *are* $\{0,1\}$ *and* $\{2\}$*, respectively. It can be verified that for every* $\lambda_1 \in \{0,1\}$ *and* $\lambda_2 \in \{2\}$*, the matrix*

$$F_{\mathrm{c}} = \begin{bmatrix} (A_{\mathrm{c}1} - \lambda_1 I_4) \\ (A_{\mathrm{c}2} - \lambda_2 I_4) \\ C_{\mathrm{c}2} \\ C_{\mathrm{c}2} \end{bmatrix}$$

is of full rank. Thus, the matrices $A_{\mathrm{c}1}$ *and* $A_{\mathrm{c}2}$ *do not have a common right eigenvector constrained by* $C_{\mathrm{c}1}$ *and* $C_{\mathrm{c}2}$*. In the similar way, one can verify that the matrices*

$$A_{\mathrm{r}1} = \begin{bmatrix} A_{1,1} & A_{1,2} \\ \mathbf{0}_{1,3} & \mathbf{0}_{1,1} \end{bmatrix}, \quad A_{\mathrm{r}2} = \begin{bmatrix} \mathbf{0}_{3,3} & \mathbf{0}_{3,1} \\ A_{2,1} & A_{2,2} \end{bmatrix}$$

do not have a common left eigenvector constrained by

$$B_{\mathrm{r}1} = \begin{bmatrix} B_1 \\ \mathbf{0}_{1,2} \end{bmatrix}, \quad B_{\mathrm{r}2} = \begin{bmatrix} \mathbf{0}_{3,2} \\ B_2 \end{bmatrix}.$$

Thus, this Roesser model cannot be reduced by directly applying the common eigenvector approach.

However, as show in Example 4.4, the given Roesser model of (5.74) *is equivalent to the following Roesser model:*

$$\bar{A} = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 2 \end{array}\right], \quad \bar{B} = \left[\begin{array}{cc} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ \hline 0 & 1 \end{array}\right], \quad \bar{C} = \left[\begin{array}{c} 2 \\ 0 \\ 2 \\ \hline 0 \end{array}\right]^{\mathrm{T}}, \quad \bar{D} = \left[\begin{array}{cc} 0 & 0 \end{array}\right]. \tag{5.76}$$

For this new equivalent Roesser model, we have

$$\bar{A}_{\mathrm{c}1} = \begin{bmatrix} \bar{A}_{1,1} & \mathbf{0}_{3,1} \\ \bar{A}_{2,1} & \mathbf{0}_{1,1} \end{bmatrix}, \quad \bar{A}_{\mathrm{c}2} = \begin{bmatrix} \mathbf{0}_{3,3} & \bar{A}_{1,2} \\ \mathbf{0}_{1,3} & \bar{A}_{2,2} \end{bmatrix}, \quad \bar{C}_{\mathrm{c}1} = \begin{bmatrix} \bar{C}_1 & \mathbf{0}_{1,1} \end{bmatrix}, \quad \bar{C}_{\mathrm{c}1} = \begin{bmatrix} \mathbf{0}_{1,3} & \bar{C}_2 \end{bmatrix}. \tag{5.77}$$

It can be verified that for the eigenvalues $\lambda_1 = 0$ *and* $\lambda_2 = 2$ *of* $\bar{A}_{\mathrm{c}1}$ *and* $\bar{A}_{\mathrm{c}2}$*, respectively, the matrix*

$$F_{\mathrm{c}} = \begin{bmatrix} (\bar{A}_{\mathrm{c}1} - \lambda_1 I_4) \\ (\bar{A}_{\mathrm{c}2} - \lambda_2 I_4) \\ \bar{C}_{\mathrm{c}2} \\ \bar{C}_{\mathrm{c}2} \end{bmatrix} \tag{5.78}$$

has rank $2 < 3$ *and then has not full rank. Thus, the matrices* $\bar{A}_{\mathrm{c}1}$ *and* $\bar{A}_{\mathrm{c}2}$ *have a common right vector, say*

$$\boldsymbol{\omega} = \left[\begin{array}{cccc} -1 & 0 & 1 & 0 \end{array}\right], \tag{5.79}$$

constrained by $\bar{C}_{\mathrm{c}1}$ *and* $\bar{C}_{\mathrm{c}2}$*, and then the Roesser model of* (5.76) *can be reduced by the proposed common eigenvector approach. Then, by the proposed reduction procedure one can obtain the following lower-order Roesser model:*

$$\hat{A} = \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 2 \end{array}\right], \quad \hat{B} = \left[\begin{array}{cc} 0 & 0 \\ 1 & 0 \\ \hline 0 & 1 \end{array}\right], \quad \hat{C} = \left[\begin{array}{c} 2 \\ \hline 0 \\ 0 \end{array}\right]^{\mathrm{T}}, \quad D = \left[\begin{array}{cc} 0 & 0 \end{array}\right], \tag{5.80}$$

with order $\hat{\boldsymbol{r}} = (2, 1)$.

Remark 5.13. *This example shows that due to the complexity of n-D reduction problem there being no further reduction via the common eigenvector approach does not guarantee the existence of a further exact reduction of a Roesser model to a smaller Roesser model with the same transfer function. The results indicate that this aspect of the problem requires further study.*

5.2.3 Comparisons to Existing Results and Further Generalization

To see more details on the effectiveness and novelty of the common eigenvector approach, in this section, we give some further comparisons to the representative exact order reduction approaches including the eigenvalue trim approach [15], the trim approach [23, 24], the elementary operation approach [32], the n-D Jordan transformation approach [29]. Then, further generalization will be given.

Comparisons to Existing Results

Lemma 5.1. *If an n-D Roesser model $(A, B, C, D; \boldsymbol{r})$ is not eigenvalue co-trim, $A_{\mathrm{c}1}, \ldots, A_{\mathrm{c}n}$ have a common right eigenvector $\boldsymbol{\omega}$ constrained by $C_{\mathrm{c}1}, \ldots, C_{\mathrm{c}1}$, however the reverse is not necessarily true. Dually, if an n-D Roesser model $(A, B, C, D; \boldsymbol{r})$ is not eigenvalue trim, matrices $A_{\mathrm{r}1}, \ldots, A_{\mathrm{r}n}$ have a common left eigenvector $\boldsymbol{\omega}$ constrained by $B_{\mathrm{r}1}, \ldots, B_{\mathrm{r}1}$, however the reverse is not necessarily true.*

Proof. The result for the constrained common left eigenvector is dual. Therefore, for simplicity, we only show the relationship between the constrained common right eigenvector and the eigenvalue co-trim.

Suppose that the n-D Roesser model $(A, B, C, D; \boldsymbol{r})$ is not eigenvalue co-trim, i.e., there are indices $i \in \{1, \ldots, n\}$ and $t \in \{1, \ldots, l_i\}$ such that the matrix $\tilde{M}_{-i}$ in (4.6) has no full column rank. Thus, there is a nonzero vector $\boldsymbol{\omega}_i$ such that $\tilde{M}_{-i}\boldsymbol{\omega}_i = \mathbf{0}$, which gives that

$$A_{i,i}\boldsymbol{\omega}_i = \lambda_{i,t}\boldsymbol{\omega}_i, \quad C_i\boldsymbol{\omega}_i = \mathbf{0},\ A_{k,i}\boldsymbol{\omega}_i = \mathbf{0} \quad \text{for} \quad \text{all } k \neq i. \tag{5.81}$$

If let $\boldsymbol{\omega} = \begin{bmatrix} \mathbf{0}_{1,r_1} & \ldots & \mathbf{0}_{1,r_{i-1}} & \boldsymbol{\omega}_i & \mathbf{0}_{1,r_{i+1}} & \ldots & \mathbf{0}_{1,r_n} \end{bmatrix}$, we can verify that

$$\begin{aligned} A_{\mathrm{c}i}\boldsymbol{\omega} =& \lambda_{i,t}\boldsymbol{\omega}, \\ A_{\mathrm{c}k}\boldsymbol{\omega} =& 0\boldsymbol{\omega} \quad \text{for} \quad \text{all } k \neq i, \\ C_{\mathrm{c}1}\boldsymbol{\omega} =& \ldots = C_{\mathrm{c}n}\boldsymbol{\omega} = \mathbf{0}, \end{aligned} \tag{5.82}$$

which shows that $A_{\mathrm{c}1}, \ldots, A_{\mathrm{c}n}$ have a common right eigenvector $\boldsymbol{\omega}$ constrained by $C_{\mathrm{c}1}$, ..., $C_{\mathrm{c}1}$. Thus, if a given n-D Roesser model $(A, B, C, D; \boldsymbol{r})$ is not eigenvalue co-trim, $A_{\mathrm{c}1}, \ldots, A_{\mathrm{c}n}$ have a common right eigenvector $\boldsymbol{\omega}$ constrained by $C_{\mathrm{c}1}, \ldots, C_{\mathrm{c}1}$.

It should be noted that for the 3-D Roesser model (5.65) in Example 5.7, matrices $A_{\rm c1}$, $A_{\rm c2}$ and $A_{\rm c3}$ have a common right eigenvector $\boldsymbol{\omega}$ constrained by $C_{\rm c1}$, …, $C_{\rm c1}$, whereas it can be verified that this model is eigenvalue co-trim. □

As discussed above, a non-eigenvalue co-trim n-D Roesser model $(A, B, C, D; \boldsymbol{r})$ always implies that matrices $A_{\rm c1}, \ldots, A_{{\rm c}n}$ have a common right eigenvector $\boldsymbol{\omega}$ satisfying

$$C_{\rm c1}\boldsymbol{\omega} = \ldots = C_{\rm c1}\boldsymbol{\omega} = \mathbf{0}.$$

However, the reverse is not true. The result for eigenvalue trim is dual. Thus, the reduction approach given in [15] is just a special case of the common eigenvector approach given here. Moreover, it has been shown in [15] the reduction approaches of [23, 24, 32] are just special cases of the eigenvalue trim approach. Therefore, the common eigenvector approach is more general and effective than the existing methods given in [15, 23, 24, 29, 32].

5.2.4 Further Generalization

It is possible to further generalize the proposed common eigenvector approach to the Roesser model with state delay. The n-D discrete linear system with state delay is in the form of (3.10) with addition term $A_{\rm d}\boldsymbol{x}_{\rm d}(i_1, \ldots, i_n)$ [81], i.e.,

$$\begin{aligned}\boldsymbol{x}'(i_1, \ldots, i_n) =& A\boldsymbol{x}(i_1, \ldots, i_n) + A_{\rm d}\boldsymbol{x}_{\rm d}(i_1, \ldots, i_n) \\ &+ B\boldsymbol{u}(i_1, \ldots, i_n)\end{aligned} \tag{5.83}$$

with

$$\boldsymbol{x}_{\rm d}(i_1, \ldots, i_n) = \begin{bmatrix} \boldsymbol{x}_1(i_1 - d, i_2, \ldots, i_n) \\ \vdots \\ \boldsymbol{x}_n(i_1, \ldots, i_{n-1}, i_n - d) \end{bmatrix}$$

. For simplicity, this Roesser model with state delay is denoted by $(A, A_{\rm d}, B, C, D; \boldsymbol{r})$. The corresponding transfer matrix is

$$H(z_1, \ldots, z_n) = CZ(I_r - AZ - A_{\rm d}Z^{d+1})^{-1}B + D$$

with

$$Z = \mathrm{diag}\{z_1 I_{r_1}, \ldots, z_n I_{r_n}\}$$

.

Then, the exact order reduction for the case with state delay can be stated as follows. For a Roesser model with state delay given by $(A, A_{\mathrm{d}}, B, C, D; \boldsymbol{r})$, find another Roesser model with state delay given by $(\hat{A}, \hat{A}_{\mathrm{d}}, \hat{B}, \hat{C}, D; \hat{\boldsymbol{r}})$ such that

$$\begin{aligned}\hat{C}\hat{Z}(I_{\hat{r}} - \hat{A}\hat{Z} - \hat{A}_{\mathrm{d}}\hat{Z}^{d+1})^{-1}\hat{B} &= CZ(I_r - AZ - A_{\mathrm{d}}Z^{d+1})^{-1} \\ \hat{r} &< r\end{aligned}$$

with $\hat{\boldsymbol{r}} = (\hat{r}_1, \ldots, \hat{r}_n)$, $\hat{r} = \hat{r}_1 + \ldots + \hat{r}_n$ and $\hat{Z} = \mathrm{diag}\{z_1 I_{\hat{r}_1}, \ldots, z_n I_{\hat{r}_n}\}$. Then, the following result can be generalized.

Lemma 5.2. *An n-D Roesser model with state delay given by* $(A, A_{\mathrm{d}}, B, C, D; \boldsymbol{r})$ *is reducible if* $A_{\mathrm{c}1}$, …, $A_{\mathrm{c}n}$, $A_{\mathrm{dc}1}$, …, $A_{\mathrm{dc}n}$ *have a common right eigenvector constrained by* $C_{\mathrm{c}1}, \ldots, C_{\mathrm{c}n}$, *where* $A_{\mathrm{c}i}$ *and* $C_{\mathrm{c}i}$ *are defined in* (5.49a), *and* $A_{\mathrm{dc}i} = E_i A_{\mathrm{dc}}$ *with* E_i *given in* (5.48), *or equivalently, the matrix*

$$F_{\mathrm{dc}} \triangleq \begin{bmatrix} (A_{\mathrm{c}1} - \lambda_1 I_r) \\ \vdots \\ (A_{\mathrm{c}n} - \lambda_n I_r) \\ (A_{\mathrm{dc}1} - \lambda_{\mathrm{d}1} I_r) \\ \vdots \\ (A_{\mathrm{dc}n} - \lambda_{\mathrm{d}n} I_r) \\ C_{\mathrm{c}1} \\ \vdots \\ C_{\mathrm{c}n} \end{bmatrix} \tag{5.84}$$

is not of full column rank for some eigenvalue λ_i *and* $\lambda_{\mathrm{d}i}$ *of* $A_{\mathrm{c}i}$ *and* $A_{\mathrm{dc}i}$, *respectively,* $i \in \{1, \ldots, n\}$.

The proof of Lemma 5.2 can be done in a similar way to the ones of Theorems 5.5 and 5.6, and thus the details are omitted here.

Remark 5.14. *The common eigenvector approach can treat the eigenvalues of both the system matrix* A *and the state-delay system matrix* A_{d}. *However, the eigenvalue trim approach given in [15] is difficult to be generalized to the case involving both* A *and* A_{d}.

5.3 Calculation of Constrained Common Eigenvectors

In this section, a Gröbner basis method [82, 83] is established to directly obtain a constrained common eigenvector without calculating the relevant eigenvalues. To this end, we introduce the following notation.

Definition 5.3. *Let $\{f_1(\boldsymbol{x}), \ldots, f_l(\boldsymbol{x})\}$ be the set generated by the union of the entries in $C_i\boldsymbol{x}$ and the 2×2 minors of the matrices $X_i = [A_i\boldsymbol{x}\ \boldsymbol{x}]$, $i = 1, \ldots, n$. Then, $\boldsymbol{I}_{A_1,\ldots,A_n;C_1,\ldots,C_n}$ is defined to be the ideal generated by $\{f_1(\boldsymbol{x}), \ldots, f_l(\boldsymbol{x})\}$.*

The basic idea to obtain a common right eigenvector of $A_1, \ldots, A_n$ constrained by $C_1, \ldots, C_n$ is as follows. We know that $\boldsymbol{x} = [x_1\ \ldots\ x_r]^{\mathrm{T}}$ is a common right eigenvector of $A_1, \ldots, A_n$ if and only if every pair of $A_i\boldsymbol{x}$ and $\boldsymbol{x}$ is linear dependent for $i \in \{1, \ldots, n\}$ [84]. Thus, a nonzero vector $\boldsymbol{x}$ is a common right eigenvector of $A_1, \ldots, A_n$ constrained by $C_1, \ldots, C_n$ if and only if $X_i = [A_i\boldsymbol{x}\ \boldsymbol{x}]$ and $C_i\boldsymbol{x}$ have rank 1 and 0, respectively, for every $i = 1, \ldots, n$. Then, we have the following results.

Theorem 5.9. *Let $\mathcal{G} = \{g_1(\boldsymbol{x}), \ldots, g_m(\boldsymbol{x})\}$ be the reduced Gröbner basis of the ideal $\boldsymbol{I}_{A_1,\ldots,A_n;C_1,\ldots,C_n}$. Then, the matrices $A_1, \ldots, A_n$ have a common right eigenvector constrained by $C_1, \ldots, C_n$ if and only if there is a nonzero vector $\boldsymbol{\omega}$ such that $g_1(\boldsymbol{\omega}) = \ldots = g_m(\boldsymbol{\omega}) = 0$.*

Proof. A nonzero vector $\boldsymbol{\omega}$ is a common right eigenvector of $A_1, \ldots, A_n$ constrained by $C_1, \ldots, C_n$ if and only if $X_i = [A\boldsymbol{x}\ \boldsymbol{x}]$ and $C_i\boldsymbol{x}$ have rank 1 and 0, respectively for all $i = 1, \ldots, n$, i.e., $f_1(\boldsymbol{\omega}) = \ldots = f_l(\boldsymbol{\omega}) = 0$ with $\langle f_1(\boldsymbol{x}), \ldots, f_l(\boldsymbol{x})\rangle = \boldsymbol{I}_{A_1,\ldots,A_n;C_1,\ldots,C_n}$. By Gröbner theory [82, 83], we known that the solutions of $g_1(\boldsymbol{\omega}) = \ldots = g_m(\boldsymbol{\omega}) = 0$ and $f_1(\boldsymbol{\omega}) = \ldots = f_l(\boldsymbol{\omega}) = 0$ are equal. Then, the proof is completed. □

Theorem 5.10. *Let $\mathcal{G} = \{g_1(\boldsymbol{x}), \ldots, g_m(\boldsymbol{x})\}$ be the reduced Gröbner basis of the ideal $\boldsymbol{I}_{A_1^{\mathrm{T}},\ldots,A_n^{\mathrm{T}};B_1^{\mathrm{T}},\ldots,B_n^{\mathrm{T}}}$. Then, the matrices $A_1, \ldots, A_n$ have a common left eigenvector constrained by $B_1, \ldots, B_n$ if and only if there is a nonzero vector $\boldsymbol{\omega}$ such that $g_1(\boldsymbol{\omega}) = \ldots = g_m(\boldsymbol{\omega}) = 0$.*

The proof is dual to the one for Theorem 5.9, and thus is omitted.

In view of Theorems 5.5 and 5.7, one can easily obtain the following result.

Theorem 5.11. *For a given n-D Roesser model* $(A, B, C, D; \boldsymbol{r})$*, let* $\mathcal{G} = \{g_1(\boldsymbol{x}), \ldots, g_m(\boldsymbol{x})\}$ *be the reduced Gröbner basis of the ideal* $\boldsymbol{I}_{A_{c1},\ldots,A_{cn};C_{c1},\ldots,C_{cn}}$ *or the ideal* $\boldsymbol{I}_{A_{r1}^{\mathrm{T}},\ldots,A_{rn}^{\mathrm{T}};B_{r1}^{\mathrm{T}},\ldots,B_{rn}^{\mathrm{T}}}$. *If there is a nonzero vector* $\boldsymbol{\omega}$ *such that* $g_1(\boldsymbol{\omega}) = \ldots = g_m(\boldsymbol{\omega}) = 0$*, then the given Roesser model is reducible.*

Remark 5.15. *We would like to remark that the order r of n-D Roesser model is equal to the number of variables in* $\boldsymbol{I}_{A_1^{\mathrm{T}},\ldots,A_n^{\mathrm{T}};B_1^{\mathrm{T}},\ldots,B_n^{\mathrm{T}}}$*, and the computation of the reduced Gröbner basis is time-consuming for a large number of variables. For this reason, a large amount of work has been done aiming to improve efficiency so that it can cope with large numbers of variables or large order r (see, i.e., [85, 86]). The reduced Gröbner basis can also be easily obtained by a computer program such as "gbasis" in MAPLE and "groebner::gbasis" in MATLAB.*

Example 5.9. *Consider the matrices* A_{c1}, A_{c2}*,* A_{c3}*,* C_{c1}, C_{c2} *and* C_{c3} *in (5.66) of the given* 3*-D Roesser model in (5.65). A reduced Gröbner basis of the ideal generated by* $I_{A_{c1},A_{c2},A_{cc};C_{c1},C_{c2},C_{c3}}$ *can be calculated as*

$$\begin{aligned}\mathcal{G} =&\{2x_1 - x_4,\ {x_2}^2 - 2x_7x_8,\ 2{x_8}^2 - 2x_7x_8 + x_2x_4,\\ &- {x_8}^2 + x_7x_8 + x_2x_7,\ x_2x_8 + x_7x_8 + {x_8}^2,\\ &x_3,\ {x_4}^2 + 4{x_8}^2, x_4x_7 - 2{x_8}^2,\ x_4x_8 + 2x_7x_8,\\ &x_5 + x_7,\ x_6 + x_8,\ {x_7}^2 + {x_8}^2,\ x_9,\ x_{10}\}.\end{aligned}$$

One nonzero solution of the polynomials given in $\mathcal{G}$ *is just the vector* $\boldsymbol{\omega}$ *in (5.67), which is common right eigenvector of* A_{c1}, A_{c2} *and* A_{c3} *constrained by* C_{c1}, C_{c2} *and* C_{c3}*. Thus, the given* 3*-D Roesser can be reduced.*

5.4 Contribution Summary

The notion of constrained common eigenvector has been introduced to simultaneously take into account the eigenvalues of multiple matrices. Based on this constrained common eigenvector, sufficient reducibility conditions for the n-D F-M model and the n-D Roesser model have been developed, which can be viewed as a kind of generalization of PBH tests for the exact reducibility of n-D state-space models. A Gröbner basis approach has also been proposed to compute such a constrained common eigenvector. Moreover, a generalization to the state delay case has been given to show this method more applicable. Examples have also been given to illustrate the details and effectiveness of the new method.

Chapter 6

Common Invariant Subspace Approach to Exact Order Reduction for State-space Models of Multidimensional Systems

In the previous chapter, the exact order reduction of n-D state-space models has been studied based on common eigenvectors. In the present chapter, we take a more general approach, i.e., the so-called common invariant subspace approach, and then can exactly reduce n-D state-space models even when the previous conditions are not met. It turns out that the new common invariant subspace can obtain a minimal state-space realization in the noncommutative setting, i.e., the variables in the system are not commutative.

Specifically, we first present some notations in the noncommutative setting. Then, a necessary and sufficient condition for the existence of a common invariant subspace is proposed. Finally, new reducibility conditions for are developed for the F-M model and the Roesser model based on a common invariant subspace.

This chapter is organized as follows. In section 6.1, we introduce n-D models in the noncommutative setting. In Section 6.2, a necessary and sufficient condition for the existence of common invariant subspace is developed. Then, based on that common invariant subspace, a new necessary and sufficient reducibility condition is presented for the n-D F-M in Section 6.3. Section 6.4 generalize these results in the F-M model to the Roesser model case. Finally, conclusions are made in Section 6.5.

6.1 Noncommutative Setting

In this section, we will introduce the n-D models in the noncommutative setting. To this end, we need the following notations.

Notation 6.1. *For a positive integer n, denote by $\mathbf{n}$ the set $\{1, \ldots, n\}$ and by $\mathcal{F}_{\mathbf{n}}$ the finite set of sequences of elements of the set $\mathbf{n}$. The elements of $\mathcal{F}_{\mathbf{n}}$ are also referred to as strings or words over the set $\mathbf{n}$ [87]. Each non-empty word w is of the form*

$$w = \alpha_1 \alpha_2 \ldots \alpha_l$$

for some $\alpha_1, \ldots, \alpha_l \in \mathbf{n}$. The element α_k is called the kth letter of w, for $k = 1, \ldots, l$ and l is called the length of w and is denoted by $|w|$. We denote by ϵ the empty (word) sequence and the length of ϵ is zero by definition. We denote by $\mathcal{F}_{\mathbf{n}}^{+}$ the set of non-empty words. Let the set $\mathcal{F}_{\mathbf{n}}^{+}(k_1, \ldots, k_n) \subset \mathcal{F}_{\mathbf{n}}^{+}$ such that the number of letters $i \in \mathbf{n}$ in the word w is equal to k_i for each word $w = \alpha_1 \alpha_2 \ldots \alpha_l \in \mathcal{F}_{\mathbf{n}}^{+}(k_1, \ldots, k_n)$.

Notation 6.2. *Let $A_i \in \mathbf{R}^{n \times n}$, $i \in \mathbf{n}$ be a finite collection of matrices indexed by elements of the set $\mathbf{n}$ and let $v = \alpha_1 \alpha_2 \ldots \alpha_l \in \mathcal{F}_{\mathbf{n}}$. The $n \times n$ matrix A_v is defined as follows.*

$$A_v = A_{\alpha_l} A_{\alpha_l - 1} \ldots A_{\alpha_1}, \tag{6.1}$$

where A_v is the identity matrix if $v = \epsilon$, i.e., $A_\epsilon = I$.

Notation 6.3. *Let*

$$A = \begin{bmatrix} A_{1,1} & A_{1,2} & \ldots & A_{1,n} \\ A_{2,1} & A_{2,2} & \ldots & A_{2,n} \\ \vdots & \vdots & & \vdots \\ A_{n,1} & A_{n,2} & \ldots & A_{n,n} \end{bmatrix} \in \mathbf{R}^{r \times r}, \tag{6.2}$$

with $A_{i,k} \in \mathbf{R}^{r_i \times r_k}$, $i, k = 1, \ldots, n$. Let $v = \alpha_1 \alpha_2 \ldots \alpha_l \in \mathcal{F}_{\mathbf{n}}$. Then, the $r_{\alpha_l} \times r_{\alpha_1}$ matrix A_{*v} is defined as follows.

$$A_{*v} = A_{\alpha_l, \alpha_{l-1}} A_{\alpha_{l-1}, \alpha_{l-2}} \ldots A_{\alpha_2, \alpha_1}, \tag{6.3}$$

where A_{*v} is the identity matrix if $|v| = 1$, i.e., $A_{*k} = I$ for $k = 1, \ldots, n$.

Notation 6.4. *A formal power series $H(z_1, \dots, z_n)$ in noncommutative setting is defined by*

$$H(z_1, \dots, z_n) = \sum_{w \in \mathcal{F}_{\mathbf{n}}} H(w) z^w, \tag{6.4}$$

where $H(w)$ is the coefficient matrix w.r.t.

$$z^w = z_{\alpha_l} z_{\alpha_{l-1}} \dots z_{\alpha_1}, \tag{6.5}$$

with

$$w = \alpha_1 \alpha_2 \dots \alpha_l.$$

Note that here the variables $z_1, \dots, z_n$ in (6.4) are not commutative, i.e.,

$$z_{\alpha_1} z_{\alpha_2} \neq z_{\alpha_2} z_{\alpha_1}$$

for all $\alpha_1 \neq \alpha_2$ and $\alpha_1, \alpha_2 \in \mathbf{n}$.

Now, we can introduce the n-D transfer function matrices in the noncommutative setting.

For the transfer function matrix $H(z_1, \dots, z_n)$ in (3.23) of the n-D F-M model $(\boldsymbol{A}, \boldsymbol{B}, C, D; r)$ in (3.22), if one assumes the unite delay operators $z_1, \dots, z_n$ are noncommutative, i.e, $z_{\alpha_1} z_{\alpha_2} \neq z_{\alpha_2} z_{\alpha_1}$ for $\alpha_1 \neq \alpha_2$ and $\alpha_1, \alpha_2 = 1 \dots, n$, then this transfer function matrix is said to be noncommutative. That is, the noncommutative transfer function matrix of the F-M model $(\boldsymbol{A}, \boldsymbol{B}, C, D; r)$ is

$$\begin{aligned} & H(z_1, \dots, z_n) \\ =& C \left(\alpha_r - \sum_{i=1}^{n} A_i z_i \right)^{-1} \left(\sum_{i=1}^{n} B_i z_i \right) + D \\ =& C \sum_{k=0}^{\infty} \left(\sum_{i=1}^{n} A_i z_i \right)^{k} \left(\sum_{i=1}^{n} B_i z_i \right) + D \\ =& \sum_{w \in \mathcal{F}_{\mathbf{n}}^{+}} C A_v B_{\alpha_1} z^w + D, \end{aligned} \tag{6.6}$$

with $w = \alpha_1 \alpha_2 \dots \alpha_l = \alpha_1 v \in \mathcal{F}_{\mathbf{n}}^{+}$, A_v defined in (6.1) and z^w defined in (6.5).

In the similar way, the noncommutative transfer function (matrix) of the Roesser model $(A, B, C, D; \boldsymbol{r})$ can be defined as

$$
\begin{aligned}
H(z_1, \ldots, z_n) &= C(I_r - ZA)^{-1} ZB + D \\
&= \begin{bmatrix} C_1 & C_2 & \ldots & C_n \end{bmatrix} \left(I_r - \begin{bmatrix} A_{1,1}z_1 & A_{1,2}z_1z_1 & \ldots & A_{1,n}z_1 \\ A_{2,1}z_2 & A_{2,2}z_2z_2 & \ldots & A_{2,n}z_2 \\ \vdots & \vdots & & \vdots \\ A_{n,1}z_n & A_{n,2}z_nz_n & \ldots & A_{n,n}z_n \end{bmatrix} \right)^{-1} \begin{bmatrix} B_1z_1 \\ B_2z_2 \\ \vdots \\ B_nz_n \end{bmatrix} \\
&= \begin{bmatrix} C_1 & C_2 & \ldots & C_n \end{bmatrix} \left(\sum_{k=0}^{\infty} \begin{bmatrix} A_{1,1}z_1 & A_{1,2}z_1z_1 & \ldots & A_{1,n}z_1 \\ A_{2,1}z_2 & A_{2,2}z_2z_2 & \ldots & A_{2,n}z_2 \\ \vdots & \vdots & & \vdots \\ A_{n,1}z_n & A_{n,2}z_nz_n & \ldots & A_{n,n}z_n \end{bmatrix}^k \right) \begin{bmatrix} B_1z_1 \\ B_2z_2 \\ \vdots \\ B_nz_n \end{bmatrix} \\
&= \sum_{w \in \mathcal{F}_{\mathbf{n}}^{+}} (C_{\alpha_l} A_{*w} B_{\alpha_1}) z^w + D,
\end{aligned} \tag{6.7}
$$

with $w = \alpha_1\alpha_2\ldots\alpha_l \in \mathcal{F}_{\mathbf{n}}^{+}$, A_{*v} defined in (6.3) and z^w defined in (6.5). Note that $C_{\alpha_l} = C_{\alpha_1}$ if $w = \alpha_1$.

Next, we define the noncommutative realization and noncommutative exact order reduction of n-D systems in the F-M model.

Definition 6.1 (The F-M Model Realization in the Noncommutative Setting)**.** *For a given formal power series in the form of (6.4) if there are matrices $A_1, \ldots, A_n$, $B_1, \ldots, B_n$, C and D such that*

$$
CA_vB_{\alpha_1} = H(w),\ D = H(\epsilon), \tag{6.8}
$$

for $w = \alpha_1\alpha_2\ldots\alpha_l = \alpha_1 v \in \mathcal{F}_{\mathbf{n}}^{+}$, with A_v defined in (6.1) *and z^w defined in* (6.5)*, the F-M model $(\mathbf{A}, \mathbf{B}, C, D; \boldsymbol{r})$ with $\mathbf{A} = (A_1, \ldots, A_n)$ and $\mathbf{B} = (B_1, \ldots, B_n)$ is said to be a realization of (6.4) in the noncommutative setting.*

Definition 6.2 (Exact Order Reduction of the F-M Model in the Noncommutative Setting)**.** *The exact order reduction to be considered for F-M state-space models in the noncommutative setting can be stated as follows: for a given F-M state-space model $(\mathbf{A}, \mathbf{B}, C, D; r)$, find another F-M model $(\hat{\mathbf{A}}, \hat{\mathbf{B}}, \hat{C}, D; \hat{r})$ such that*

$$
\hat{C}\hat{A}_v\hat{B}_{i_1} z^w = CA_vB_{\alpha_1} z^w,\ \hat{r} \le r, \tag{6.9}
$$

for all with $w = \alpha_1\alpha_2\ldots\alpha_l = \alpha_1 v \in \mathcal{F}_{\mathbf{n}}^{+}$, with A_v defined in (6.1) *and z^w defined in* (6.5).

Definition 6.3 (Minimal F-M Realization in the Noncommutative Setting)**.** *An n-D Roesser model* $(\hat{A}, \hat{B}, \hat{C}, D; \hat{\boldsymbol{r}})$ *of a noncommutative transfer matrix* $H(z_1, \ldots, z_n)$ *is a minimal noncommutative realization if* $\hat{r}$ *is lowest among all the noncommutative realizations of* $H(z_1, \ldots, z_n)$*.*

In the similar way, we define the realization and exact order reduction of n-D systems in the Roesser model in the noncommutative setting.

Definition 6.4 (The Roesser Model Realization in the Noncommutative Setting)**.** *For a given formal power series in the form of (6.4) if there are matrices A, B C and D in (3.12) such that*

$$C_{\alpha_l} A_{*v} B_{\alpha_1} = H(w), \tag{6.10a}$$

$$D = H(\epsilon), \tag{6.10b}$$

for all $w = \alpha_1\alpha_2\ldots\alpha_l = \alpha_1 v \in \mathcal{F}_{\mathbf{n}}^{+}$*, the Roesser model* $(A, B, C, D; \boldsymbol{r})$ *is said to be a realization of (6.4) in the noncommutative setting.*

Definition 6.5 (Exact Order Reduction of the Roesser Model in the Noncommutative Setting)**.** *The exact order reduction to be considered for Roesser state-space models in the noncommutative setting can be stated as follows: for a given Roesser state-space model* $(A, B, C, D; \boldsymbol{r})$*, find another Roesser model* $(\hat{A}, \hat{B}, \hat{C}, D; \hat{\boldsymbol{r}})$ *such that*

$$\hat{C}_{\alpha_l} \hat{A}_{*w} \hat{B}_{\alpha_1} = C_{\alpha_l} A_{*w} B_{\alpha_1}, \tag{6.11a}$$

$$\hat{r} \leq r, \tag{6.11b}$$

for all $w = \alpha_1\alpha_2\ldots\alpha_l \in \mathcal{F}_{\mathbf{n}}^{+}$*, with* A_{*v} *defined in* (6.3) *and* z^w *defined in* (6.5)*.*

Definition 6.6 (Minimal Roesser Model Realization in the Noncommutative Setting)**.** *An n-D Roesser model* $(\hat{A}, \hat{B}, \hat{C}, D; \hat{\boldsymbol{r}})$ *of a noncommutative transfer matrix* $H(z_1, \ldots, z_n)$ *is a minimal noncommutative realization if* $\hat{r}$ *is lowest among all the noncommutative realizations of* $H(z_1, \ldots, z_n)$*.*

Remark 6.1. *It should be noted that the realization and exact order reduction in the noncommutative setting are just a special cases for general setting, which will also be discussed in the next chapter.*

6.2 Necessary and Sufficient Conditions for the Existence of Common Invariant Subspace

In this section, a necessary and sufficient condition for the existence of common invariant subspace is proposed, which plays an important roll in the derivation of the main results for exact order reduction of the n-D F-M models and n-D Roesser models and also provides a way to compute a common invariant subspace. To this end, we introduced the following notations.

Notation 6.5. *For given sets $\boldsymbol{A} = \{A_1, \ldots, A_n\}$ and $\boldsymbol{C} = \{C_1, \ldots, C_n\}$ with $A_i \in \mathbf{R}^{r\times r}$ and $C_i \in \mathbf{R}^{p_i\times r}$, the matrices $\mathcal{N}_{k,i}(\boldsymbol{A};\boldsymbol{C})$ and $\mathcal{N}_k(\boldsymbol{A};\boldsymbol{C})$ are defined as*

$$\mathcal{N}_{k,i}(\boldsymbol{A};\boldsymbol{C}) \triangleq \mathcal{N}_{k-1}A_i, \tag{6.12a}$$

$$\mathcal{N}_k(\boldsymbol{A};\boldsymbol{C}) \triangleq \begin{bmatrix} \mathcal{N}_{k,1}(\boldsymbol{A};\boldsymbol{C}) \\ \mathcal{N}_{k,2}(\boldsymbol{A};\boldsymbol{C}) \\ \vdots \\ \mathcal{N}_{k,n}(\boldsymbol{A};\boldsymbol{C}) \end{bmatrix} = \begin{bmatrix} \mathcal{N}_{k-1}(\boldsymbol{A};\boldsymbol{C})A_1 \\ \mathcal{N}_{k-1}(\boldsymbol{A};\boldsymbol{C})A_2 \\ \vdots \\ \mathcal{N}_{k-1}(\boldsymbol{A};\boldsymbol{C})A_n \end{bmatrix} \in \mathbf{R}^{pn^{k-1}\times r}, \tag{6.12b}$$

with $\mathcal{N}_{1,i}(\boldsymbol{A};\boldsymbol{C}) = C_i$ and $r = r_1 + \ldots r_n$.

To illustrate this definition, see the following example.

Example 6.1. *For $n = 2$, one obtains:*

$$\begin{aligned}
&\mathcal{N}_{1,1}(\boldsymbol{A};\boldsymbol{C}) = C_1, \quad \mathcal{N}_{1,1}(\boldsymbol{A};\boldsymbol{C}) = C_2, \quad \mathcal{N}_1(\boldsymbol{A};\boldsymbol{C}) = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}, \\
&\mathcal{N}_{2,1}(\boldsymbol{A};\boldsymbol{C}) = CA_1, \quad \mathcal{N}_{2,1}(\boldsymbol{A};\boldsymbol{C}) = CA_2, \quad \mathcal{N}_2(\boldsymbol{A};\boldsymbol{C}) = \begin{bmatrix} CA_1 \\ CA_2 \end{bmatrix}, \\
&\mathcal{N}_{3,1}(\boldsymbol{A};\boldsymbol{C}) = \begin{bmatrix} CA_1^2 \\ CA_2A_1 \end{bmatrix}, \quad \mathcal{N}_{3,2}(\boldsymbol{A};\boldsymbol{C}) = \begin{bmatrix} CA_1A_2 \\ CA_2^2 \end{bmatrix}, \quad \mathcal{N}_3(\boldsymbol{A};\boldsymbol{C}) = \begin{bmatrix} CA_1^2 \\ CA_2A_1 \\ CA_1A_2 \\ CA_2^2 \end{bmatrix},
\end{aligned} \tag{6.13}$$

with

$$C = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}. \tag{6.14}$$

Definition 6.7. *For given sets $\boldsymbol{A} = \{A_1, \ldots, A_n\}$, $\boldsymbol{B} = \{B_1, \ldots, B_n\}$ and $\boldsymbol{C} = \{C_1, \ldots, C_n\}$, with $A_1, \ldots, A_n \in \mathbf{R}^{r\times r}$, $B_i \in \mathbf{R}^{r\times q}$, and $C_i \in \mathbf{R}^{p_i\times r}$, $i = 1, \ldots, n$, the matrix $\mathcal{O}_{k,i}(\boldsymbol{A};\boldsymbol{C})$,*

$\mathcal{O}_k(\boldsymbol{A};\boldsymbol{C})$, $\mathcal{R}_{k,i}(\boldsymbol{A};\boldsymbol{C})$ and $\mathcal{R}_k(\boldsymbol{A};\boldsymbol{C})$ are defined as:

$$\mathcal{O}_{k,i}(\boldsymbol{A};\boldsymbol{C}) = \begin{bmatrix} \mathcal{N}_{1,i}(\boldsymbol{A};\boldsymbol{C}) \\ \vdots \\ \mathcal{N}_{k,i}(\boldsymbol{A};\boldsymbol{C}) \end{bmatrix}, \tag{6.15a}$$

$$\mathcal{O}_{k}(\boldsymbol{A};\boldsymbol{C}) = \begin{bmatrix} \mathcal{N}_{1}(\boldsymbol{A};\boldsymbol{C}) \\ \vdots \\ \mathcal{N}_{k}(\boldsymbol{A};\boldsymbol{C}) \end{bmatrix}, \tag{6.15b}$$

$$\mathcal{R}^{\mathrm{T}}_{k,i}(\boldsymbol{A};\boldsymbol{B}) = \begin{bmatrix} \mathcal{N}^{\mathrm{T}}_{1,i}(\boldsymbol{A};\boldsymbol{B}) \\ \vdots \\ \mathcal{N}^{\mathrm{T}}_{k,i}(\boldsymbol{A};\boldsymbol{B}) \end{bmatrix}^{\mathrm{T}}, \tag{6.15c}$$

$$\mathcal{R}^{\mathrm{T}}_{k}(\boldsymbol{A};\boldsymbol{B}) = \begin{bmatrix} \mathcal{N}^{\mathrm{T}}_{1}(\boldsymbol{A};\boldsymbol{B}) \\ \vdots \\ \mathcal{N}^{\mathrm{T}}_{k}(\boldsymbol{A};\boldsymbol{B}) \end{bmatrix}^{\mathrm{T}}. \tag{6.15d}$$

Example 6.2. *For $n = 2$, one obtains:*

$$\mathcal{O}_{3,1}(\boldsymbol{A},\boldsymbol{C}) = \begin{bmatrix} C_1 \\ CA_1 \\ CA_1^2 \\ CA_2A_1 \end{bmatrix}, \ \mathcal{O}_{3,2}(\boldsymbol{A},\boldsymbol{C}) = \begin{bmatrix} C_2 \\ CA_2 \\ CA_1A_2 \\ CA_2A_2 \end{bmatrix}, \ \mathcal{O}_{3}(\boldsymbol{A},\boldsymbol{C}) = \begin{bmatrix} C \\ CA_1 \\ CA_2 \\ CA_1^2 \\ CA_2A_1 \\ CA_1A_2 \\ CA_2^2 \end{bmatrix}, \tag{6.16}$$

$$\mathcal{R}_{3,1}(\boldsymbol{A},\boldsymbol{B}) = \begin{bmatrix} B_1 & A_1B & A_1^2B & A_1A_2B \end{bmatrix},$$

$$\mathcal{R}_{3,2}(\boldsymbol{A},\boldsymbol{B}) = \begin{bmatrix} B_2 & A_2B & A_2A_1B & A_2A_2B \end{bmatrix},$$

$$\mathcal{R}_{3}(\boldsymbol{A},\boldsymbol{B}) = \begin{bmatrix} B & A_1B & A_2B & A_1^2B & A_1A_2B & A_2A_1B & A_2^2B \end{bmatrix},$$

with

$$C = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 & B_2 \end{bmatrix}. \tag{6.17}$$

Notation 6.6 (Lexicographic Ordering)**.** *Recall that $\mathbf{n} \triangleq \{1,\ldots,n\}$. We define a lexicographic ordering $<$ on the set $\mathcal{F}_{\mathbf{n}}$ as follows. For any $v_1, v_1 \in \mathcal{F}_{\mathbf{n}}$ with $v_1 = \alpha_1\,\alpha_2\,\ldots\,\alpha_{l_1}$ and $v_2 = \beta_1\,\beta_2\,\ldots\,\beta_{l_2}$, $v_1 < v_2$ if either $|v_1| < |v_2|$, i.e., $l_1 = l_2$ or $0 < |v_1| = |v_2|$, $v_1 \neq v_2$ and for some $k \in \{1,\ldots,|v_1|\}$, $\alpha_k < \beta_k$ with the usual ordering of integers and $\alpha_i = \beta_i$ for $i = 1,\ldots,k-1$. Note that $<$ is a complete ordering and*

$$\mathcal{F}_{\mathbf{n}} = \{v_1,\ v_2,\ldots\}, \tag{6.18}$$

with $v_1 < v_2 < \ldots$. *Note that* $v_1 = \epsilon$ *and for all* $0 < i \in \mathbf{N}$, $k \in \{1, \ldots, n\}$, $v_i < v_i\, k$.

Example 6.3. *For* $n = 3$, *we have*

$$\begin{aligned}\mathcal{F}_{\mathbf{3}} &= \left\{\, \epsilon,\;\; v_1,\, v_2,\, v_3,\, v_4,\, v_5,\, v_6,\, v_7,\, v_8,\, v_9,\, v_{10},\, v_{11},\, v_{12},\, v_{13},\, v_{14},\, \ldots \right\} \\ &= \left\{\, \epsilon,\;\; 1,\; 2,\; 3,\; 1\,1,\; 1\,2,\; 1\,3,\; 2\,1,\; 2\;\;2,\; 2\,3,\; 3\,1,\; 3\,2,\; 3\,3,\; \ldots \right\}. \end{aligned} \tag{6.19}$$

Now, we have the following main results.

Theorem 6.1. *Matrices* $A_1, \ldots, A_{\mathbf{n}} \in \mathbf{R}^{r\times r}$ *have a common right invariant subspace* $\mathcal{W}$ *such that*

$$C\boldsymbol{w} = \mathbf{0}, \tag{6.20a}$$

$$0 < \dim(\mathcal{W}) = \tilde{r}, \tag{6.20b}$$

for all $\boldsymbol{w} \in \mathcal{W}$ *with* $C_i \in \mathbf{R}^{p_i\times r}$, $i = 1, \ldots, n$, *if and only if the infinite matrix* $\mathcal{O}_\infty(\boldsymbol{A};\boldsymbol{C})$ *is rank deficient. Moreover,*

$$\mathcal{W} = \ker \mathcal{O}_\infty(\boldsymbol{A};\boldsymbol{C}). \tag{6.21}$$

Proof. (Sufficiency.) Suppose that matrices $A_1, \ldots, A_n \in \mathbf{R}^{r\times r}$ have a common right invariant subspace $\mathcal{W}$ such that (6.20) holds. Let $\{\boldsymbol{w}, \ldots, \boldsymbol{w}_{\tilde{r}}\}$ be any basis of $\mathcal{W}$ and set

$$W = \left[\; \boldsymbol{w}_1 \quad \ldots \quad \boldsymbol{w}_{\tilde{r}} \;\right]. \tag{6.22}$$

In view of the definition of common invariant subspace, we have that

$$\mathcal{O}_\infty(\boldsymbol{A};\boldsymbol{C})W = 0, \tag{6.23}$$

which imply that the rank deficiency of

$$\mathcal{O}_\infty(\boldsymbol{A};\boldsymbol{C}). \tag{6.24}$$

(Sufficiency.) Suppose the matrix $\mathcal{O}_\infty(\boldsymbol{A};\boldsymbol{C})$ is rank deficient. Let

$$\mathcal{W} \triangleq \operatorname{span}\{\boldsymbol{w}_1, \ldots, \boldsymbol{w}_{\tilde{r}}\} \triangleq \ker \mathcal{O}_\infty(\boldsymbol{A};\boldsymbol{C}). \tag{6.25}$$

We find that for any $\boldsymbol{w} \in \mathcal{W}$,

$$C\boldsymbol{w} = \mathcal{O}_1(\boldsymbol{A}; \boldsymbol{C})\boldsymbol{w} = \mathbf{0}, \tag{6.26}$$

and

$$\mathcal{O}_\infty(\boldsymbol{A}; \boldsymbol{C})\boldsymbol{A_i}\boldsymbol{w} = \mathbf{0}, i = 1, \dots, n, \tag{6.27}$$

which indicates that $A_i\boldsymbol{w} \in \mathcal{W}$. Thus, $\mathcal{W}$ is a common right invariant subspace of $A_1, \dots, A_n$ such that $C\boldsymbol{w} = \mathbf{0}$ for all $\boldsymbol{w} \in \mathcal{W}$. □

Theorem 6.2. *Matrices $A_1, \dots, A_n \in \mathbf{R}^{r\times r}$ have a common right invariant subspace $\mathcal{W}$ such that* (6.20) *holds if and only if the finite matrix $\mathcal{O}_r(\boldsymbol{A}; \boldsymbol{C})$ is rank deficient. Moreover,*

$$\mathcal{W} = \ker \mathcal{O}_r(\boldsymbol{A}; \boldsymbol{C}). \tag{6.28}$$

Proof. The proof can be completed by Theorem 6.1 and the following Lemma. □

Lemma 6.1. *The infinite matrix $\mathcal{O}_\infty(\boldsymbol{A}; \boldsymbol{C})$ with $A_i \in \mathbf{R}^{r\times r}$, $i = 1, \dots, n$ and $C_i \in \mathbf{R}^{p_i\times r}$, $i = 1, \dots, n$, is rank deficient if and only the finite matrix $\mathcal{O}_r(\boldsymbol{A}; \boldsymbol{C})$ is rank deficient.*

Proof. Omitted. □

Dually, the results on the common left invariant subspace can be obtained.

Theorem 6.3. *Matrices $A_1, \dots, A_n \in \mathbf{R}^{r\times r}$ have a common left invariant subspace $\mathcal{W}$ such that*

$$\boldsymbol{w}^{\mathrm{T}} B_i = \mathbf{0}, \tag{6.29a}$$

$$0 < \dim(\mathcal{W}) = \tilde{r}, \tag{6.29b}$$

for all $\boldsymbol{w} \in \mathcal{W}$ with $B_i \in\in \mathbf{R}^{r\times q_i}$ if and only if the infinite matrix $\mathcal{O}_\infty(\boldsymbol{A}; \boldsymbol{B})$ is rank deficient. Moreover,

$$\mathcal{W} = \ker \mathcal{O}_\infty^{\mathrm{T}}(\boldsymbol{A}; \boldsymbol{B}). \tag{6.30}$$

Theorem 6.4. *Matrices* $A_1, \ldots, A_n \in \mathbf{R}^{r\times r}$ *have a common left invariant subspace* $\mathcal{W}$ *such that such that* (6.29) *holds true if and only the finite matrix* $\mathcal{R}_r(\boldsymbol{A}; \boldsymbol{B})$ *is rank deficient. Moreover,*

$$\mathcal{W} = \ker \mathcal{R}_r^{\mathrm{T}}(\boldsymbol{A}; \boldsymbol{B}). \tag{6.31}$$

The proofs of Theorems 6.3 and 6.4 can be done in a similar way to the ones of Theorems 6.1 and 6.2, respectively. Thus the details are omitted here for brevity.

6.3 Reduction of the F-M model with Common Invariant Subspace

In this section, a new order reduction approach for the n-D F-M model based on common invariant subspace will be proposed. Next, a basic procedure will be given to exactly reduce the order of the n-D F-M model.

Theorem 6.5. *For a given n-D F-M model* $(\boldsymbol{A}, \boldsymbol{B}, C, D, r)$*, if the state matrices* $A_1, \ldots, A_n$ *have a common right invariant subspace* $\mathcal{V}$ *with a basis* $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{\tilde{r}}\}$ *satisfying*

$$C\boldsymbol{v}_k = \mathbf{0}, \quad k = 1, \ldots, \tilde{r}, \tag{6.32a}$$

$$0 < \tilde{r}, \tag{6.32b}$$

or if the state matrices $A_1, \ldots, A_n$ *have a common left invariant subspace* $\mathcal{W}$ *with a basis* $\{\boldsymbol{w}_1, \ldots, \boldsymbol{w}_{\tilde{r}}\}$ *satisfying*

$$\boldsymbol{w}^{\mathrm{T}} B_k = \mathbf{0}, \ k = 1, \ldots, n \tag{6.33a}$$

$$0 < \hat{r}, \tag{6.33b}$$

then the given n-D F-M model is reducible.

Proof. Suppose that $A_1, \ldots, A_n$ have a common right invariant subspace $\mathcal{W}$ such that (6.32) holds. (The proof for the common left invariant subspace $\mathcal{V}$ satisfying (6.32) is essentially the same, and therefore is not presented.)

Select any $r-\tilde{r}$ linearly independent vectors $\boldsymbol{v}_{\tilde{r}+1},\ldots,\boldsymbol{v}_r$ such that the matrix

$$T \triangleq \left[\begin{array}{ccc|ccc} \boldsymbol{v}_1 & \cdots & \boldsymbol{v}_{\tilde{r}} & \boldsymbol{v}_{\tilde{r}+1} & \cdots & \boldsymbol{v}_r \end{array}\right] \triangleq \left[\begin{array}{c|c} T_1 & T_2 \end{array}\right] \tag{6.34}$$

is nonsingular. Let

$$L \triangleq \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} \triangleq T^{-1}, \tag{6.35}$$

with $L_1 \in \mathbf{R}^{\tilde{r}\times r}$ and $L_2 \in \mathbf{R}^{(r-\tilde{r})\times r}$. Since $\mathcal{V}$ with a basis $\{\boldsymbol{v}_1,\ldots,\boldsymbol{v}_{\tilde{r}}\}$ is a common right invariant subspace of $A_1,\ldots,A_n$, we have that $A_i v_k$ can be expressed as a linear combination of vectors $\boldsymbol{v}_1,\ldots,\boldsymbol{v}_{\tilde{r}}$.

Therefore, there exist matrices $A_{i,1}$, $A_{i,3}$ and $A_{i,4}$ such that

$$\begin{aligned} A_iT =& A_i\left[\begin{array}{ccc|ccc} \boldsymbol{v}_1 & \cdots & \boldsymbol{v}_{\tilde{r}} & \boldsymbol{v}_{\tilde{r}+1} & \cdots & \boldsymbol{v}_r \end{array}\right] \\ =& \left[\begin{array}{ccc|ccc} A_i\boldsymbol{v}_1 & \cdots & A_i\boldsymbol{v}_{\tilde{r}} & A_i\boldsymbol{v}_{\tilde{r}+1} & \cdots & A_i\boldsymbol{v}_r \end{array}\right] \\ =& \left[\begin{array}{ccc|ccc} \boldsymbol{v}_1 & \cdots & \boldsymbol{v}_{\tilde{r}} & \boldsymbol{v}_{\tilde{r}+1} & \cdots & \boldsymbol{v}_r \end{array}\right]\begin{bmatrix} A_{i,1} & A_{i,2} \\ \mathbf{0} & A_{i,4} \end{bmatrix} \\ =& T\left[\begin{array}{c|c} A_{i,1} & A_{i,2} \\ \hline \mathbf{0} & A_{i,4} \end{array}\right], \end{aligned} \tag{6.36}$$

and then

$$\begin{aligned} LA_iT &= T^{-1}A_iT \\ &= \begin{bmatrix} A_{i,1} & A_{i,2} \\ \mathbf{0} & A_{i,4} \end{bmatrix}, \end{aligned} \tag{6.37}$$

with

$$A_{i,1} = L_1A_iR_1, \tag{6.38a}$$

$$A_{i,2} = L_1A_iA_iR_2, \tag{6.38b}$$

$$A_{i,4} = L_2A_iR_2. \tag{6.38c}$$

It follows from (6.35) that

$$LB_i = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} B_i = \begin{bmatrix} L_1B_i \\ L_2B_i \end{bmatrix} \triangleq \begin{bmatrix} B_{i,1} \\ B_{i,2} \end{bmatrix}. \tag{6.39}$$

Therefore, we have

$$
\begin{aligned}
& C\left(I_r-\sum_{i=1}^{n} z_i A_i\right)^{-1}\left(\sum_{i=1}^{n} z_i B_i\right)=CTL\left(I_r-\sum_{i=1}^{n} z_i A_i\right)^{-1} TL\left(\sum_{i=1}^{n} z_i B_i\right) \\
=& CT\left(I_r-\sum_{i=1}^{n} z_i T^{-1} A_i L^{-1}\right)^{-1}\left(\sum_{i=1}^{n} z_i LB_i\right)=CT\left(I_r-\sum_{i=1}^{n} z_i LA_i T\right)^{-1}\left(\sum_{i=1}^{n} z_i LB_i\right) \\
=& \begin{bmatrix} \mathbf{0} & C_{1,2} \end{bmatrix}\left(I_r-\sum_{i=1}^{n} z_i \begin{bmatrix} A_{i,1} & A_{i,2} \\ \mathbf{0} & A_{i,4} \end{bmatrix}\right)^{-1}\left(\sum_{i=1}^{n} z_i \begin{bmatrix} B_{i,1} \\ B_{i,2} \end{bmatrix}\right) \\
=& \begin{bmatrix} \mathbf{0} & C_{1,2} \end{bmatrix}\left(I_r-\sum_{i=1}^{n} \begin{bmatrix} z_i A_{i,1} & z_i A_{i,2} \\ \mathbf{0} & z_i A_{i,4} \end{bmatrix}\right)^{-1}\left(\sum_{i=1}^{n} \begin{bmatrix} z_i B_{i,1} \\ z_i B_{i,2} \end{bmatrix}\right) \\
=& C_{1,2}\left(I_{\hat{r}}-\sum_{i=1}^{n} z_i A_{1,4}\right)^{-1}\left(\sum_{i=1}^{n} z_i B_{i,2}\right),
\end{aligned} \tag{6.40}
$$

and $\hat{r} = r - \tilde{r} < r$. That is to say, we have obtained a new low-order n-D F-M model $(\hat{\boldsymbol{A}}, \hat{\boldsymbol{B}}, \hat{C}; \hat{r})$ with

$$\hat{A}_i = A_{i,4} = LA_iT, \quad \hat{B}_i = B_{i,2} = LB_i, \quad \hat{C} = C_{1,2} = CT.$$

$\square$

Remark 6.2. *For given n-D F-M model* $(\boldsymbol{A}, \boldsymbol{B}, C, D; r)$*, if the state matrices* $A_1, \ldots, A_n$ *have a common left invariant subspace* $\mathcal{W}$ *with a basis* $\{\boldsymbol{w}_1, \ldots, \boldsymbol{w}_{\tilde{r}}\}$ *satisfying* (6.33)*, then one selects any linearly independent vectors* $\boldsymbol{w}_{\tilde{r}+1}, \ldots, \boldsymbol{w}_r$ *such that the matrix*

$$T = \begin{bmatrix} \boldsymbol{w}_1^{\mathrm{T}} \\ \vdots \\ \boldsymbol{w}_{\tilde{r}}^{\mathrm{T}} \\ \hline \boldsymbol{w}_{\tilde{r}+1}^{\mathrm{T}} \\ \vdots \\ \boldsymbol{w}_r^{\mathrm{T}} \end{bmatrix} = \begin{bmatrix} T_1 \\ \hline T_2 \end{bmatrix} \tag{6.41}$$

is nonsingular.

Setting

$$R \triangleq \begin{bmatrix} R_1 & R_2 \end{bmatrix} \triangleq T^{-1}, \tag{6.42}$$

it can be verified that the given n-D Roesser model can be exactly reduced to the new n-D

F-M model $(\hat{\boldsymbol{A}}, \hat{\boldsymbol{B}}, \hat{C}; \hat{r})$ *with*

$$\hat{A}_i = T_2 A_i R_2, \quad \hat{B}_i = T_2 B_i, \quad \hat{C} = C_{1,2} = C R_2. \tag{6.43}$$

We now state and prove the necessity condition.

Theorem 6.6. *For a given n-D F-M model* $(\boldsymbol{A}, \boldsymbol{B}, C, D, r)$, *if there is a reduced F-M model* $(\hat{\boldsymbol{A}}, \hat{\boldsymbol{B}}, \hat{C}, \hat{D}, \hat{r})$, *then there is a common left invariant subspace* $\mathcal{V}$ *of* $A_1, \ldots, A_n$ *satisfying*

$$\boldsymbol{v}^{\mathrm{T}} B_i = \boldsymbol{0},\ k = 1, \ldots, n, \tag{6.44a}$$

$$0 < \dim(\mathcal{V}) = \hat{r} < r, \tag{6.44b}$$

for every $\boldsymbol{v} \in \mathcal{V}$, *or there is a common right invariant subspace* $\mathcal{W}$ *satisfying*

$$C\boldsymbol{w} = \boldsymbol{0}, \tag{6.45a}$$

$$0 < \dim(\mathcal{W}) = \tilde{r}, \tag{6.45b}$$

for every $\boldsymbol{w} \in \mathcal{W}$.

Proof. Suppose that there is a reduced F-M model $(\hat{\boldsymbol{A}}, \hat{\boldsymbol{B}}, \hat{C}, \hat{D}, \hat{r})$ for the given n-D F-M model $(\boldsymbol{A}, \boldsymbol{B}, C, D, r)$, but there is no common left invariant subspace $\mathcal{V}$ of $A_1, \ldots, A_n$ satisfying and there is no common right invariant subspace $\mathcal{V}$ satisfying for every $\boldsymbol{w} \in \mathcal{W}$. Then, with

$$B \triangleq \begin{bmatrix} B_1 & \ldots & B_n \end{bmatrix}, \tag{6.46a}$$

$$\hat{B} \triangleq \begin{bmatrix} \hat{B}_1 & \ldots & \hat{B}_n \end{bmatrix}, \tag{6.46b}$$

$$\boldsymbol{C} \triangleq \{C_1, C_2, \ldots, C_n\}, \tag{6.46c}$$

$$C_2 \triangleq \ldots \triangleq C_n \triangleq \mathbf{R}^{0 \times r}, \tag{6.46d}$$

$$\hat{\boldsymbol{C}} \triangleq \{\hat{C}_1, \hat{C}_2, \ldots, \hat{C}_n\}, \tag{6.46e}$$

$$\hat{C}_2 \triangleq \ldots \hat{C}_n \triangleq \mathbf{R}^{0 \times \hat{r}}, \tag{6.46f}$$

we have

$$
\begin{aligned}
&\mathcal{O}_r(\boldsymbol{A};\boldsymbol{C})\mathcal{R}_r(\boldsymbol{A};\boldsymbol{B})\\
=&\begin{bmatrix} CA_{v_1}B & CA_{v_2}B & \dots & CA_lB \\ CA_{v_2}B & CA_{v_3}B & \dots & CA_{v_{l+1}}B \\ \vdots & \dots & \ddots & \vdots \\ CA_{v_l}B & CA_{v_{l+1}}B & \dots & CA_{v_{2l-1}}B \end{bmatrix} \qquad (6.47)\\
=&\begin{bmatrix} \hat{C}\hat{A}_{v_1}\hat{B} & \hat{C}\hat{A}_{v_2}\hat{B} & \dots & \hat{C}\hat{A}_l\hat{B} \\ \hat{\hat{C}}\hat{A}_{v_2}\hat{B} & \hat{C}\hat{A}_{v_3}\hat{B} & \dots & \hat{C}\hat{A}_{v_{l+1}}\hat{B} \\ \vdots & \dots & \ddots & \vdots \\ \hat{\hat{C}}\hat{A}_{v_l}\hat{B} & \hat{C}\hat{A}_{v_{l+1}}\hat{B} & \dots & \hat{C}\hat{A}_{v_{2l-1}}\hat{B} \end{bmatrix},\\
=&\mathcal{O}_r(\hat{A}_1,\dots,\hat{A}_n;\hat{C}),\mathcal{R}_r(\hat{\boldsymbol{A}};\hat{\boldsymbol{B}}), \qquad (6.48)
\end{aligned}
$$

and then

$$
\begin{aligned}
&\operatorname{rank}\left(\mathcal{O}_r(\boldsymbol{A};\boldsymbol{C})\mathcal{R}_r(\boldsymbol{A};\boldsymbol{B})\right)\\
=&\operatorname{rank}\left(\mathcal{O}_r(\hat{A}_1,\dots,\hat{A}_n;\hat{C})\mathcal{R}_r(\hat{\boldsymbol{A}};\hat{\boldsymbol{B}})\right), \qquad (6.49)
\end{aligned}
$$

where $l=\sum_{i=1}^{k} n^{r-1}$. Since there is no common left invariant subspace $\mathcal{V}$ of $A_1,\dots,A_n$ satisfying (6.44) and there is no common right invariant subspace $\mathcal{V}$ satisfying (6.44) for every $\boldsymbol{w}\in\mathcal{W}$, we have

$$\operatorname{rank}\left(\mathcal{O}_r(A_1,\dots,A_n;C)\right)=r, \qquad (6.50a)$$

$$\operatorname{rank}\left(\mathcal{R}_r(\boldsymbol{A};\boldsymbol{B})\right)=r, \qquad (6.50b)$$

and then

$$\operatorname{rank}\left(\mathcal{O}_r(\boldsymbol{A};\boldsymbol{C})\mathcal{R}_r(\boldsymbol{A};\boldsymbol{B})\right)=r. \qquad (6.51)$$

Noting that the number of columns and the number of rows of $\mathcal{O}_r(\hat{A}_1,\dots,\hat{A}_n;\hat{C})$ and $\mathcal{R}_r(\hat{\boldsymbol{A}},\hat{\boldsymbol{B}})$ are the same $\hat{r}$, we have

$$\operatorname{rank}\left(\mathcal{O}_r(\hat{A}_1,\dots,\hat{A}_n;\hat{C})\right)\le\hat{r}, \qquad (6.52a)$$

$$\operatorname{rank}\left(\mathcal{R}_r(\hat{\boldsymbol{A}};\hat{\boldsymbol{B}})\right)\le\hat{r}. \qquad (6.52b)$$

In view of (6.49) and (6.52), we have

$$\begin{aligned}
&\operatorname{rank}\left(\mathcal{O}_r(\boldsymbol{A};\boldsymbol{C})\mathcal{R}_r(\boldsymbol{A};\boldsymbol{B})\right)\\
=&\operatorname{rank}\left(\mathcal{O}_r(\hat{A}_1,\ldots,\hat{A}_n;\hat{C})\mathcal{R}_r(\hat{\boldsymbol{A}};\hat{\boldsymbol{B}})\right)\\
\leq&\{\operatorname{rank}\left(\mathcal{O}_r(\hat{A}_1,\ldots,\hat{A}_n;\hat{C})\right),\operatorname{rank}\left(\mathcal{R}_r(\hat{\boldsymbol{A}};\hat{\boldsymbol{B}})\right)\}\leq\hat{r},
\end{aligned}\tag{6.53a}$$

which is a contradiction, since $\operatorname{rank}\left(\mathcal{O}_r(\boldsymbol{A};\boldsymbol{C})\mathcal{R}_r(\boldsymbol{A};\boldsymbol{B})\right)=r$. □

Now, two basic procedures for exactly reducing the order of a given F-M model can be given based common right invariant subspace and left invariant subspace, respectively.

Procedure 6.1: Exact Order Reduction of an n-D F-M Model Using a Common Right Invariant Subspace

---

Input : A given n-D F-M model $(\boldsymbol{A},\boldsymbol{B},C,D;r)$;

Output: A reduced-order n-D F-M model $(\hat{\boldsymbol{A}},\hat{\boldsymbol{B}},\hat{C},\hat{D};\hat{r})$;

1 *Step 1*: Find a common right invariant subspace $\mathcal{V}$ of $A_1,\ldots,A_n$ with basis vectors $\boldsymbol{v}_1,\cdots,\boldsymbol{v}_{\hat{r}}$ satisfying (6.32);

2 *Step 2*: Construct a nonsingular matrix T in the form of (6.34) and setting L in the form of (6.35);

3 *Step 3*: Obtain a reduced-order F-M model $(\hat{\boldsymbol{A}},\hat{\boldsymbol{B}},\hat{C},D;\hat{r})$ by

$$\hat{A}_i = L_2A_iR_2, \tag{6.54}$$

$$\hat{B}_i = L_2B_k, \tag{6.55}$$

$$\hat{C} = CR_2, \tag{6.56}$$

with T_2 in (6.34) and R_2 in (6.35) ;

4 **return** $(\hat{\boldsymbol{A}},\hat{\boldsymbol{B}},\hat{C},\hat{D};\hat{r})$.

Two examples will be given below to illustrate the details and effectiveness of the proposed exact order reduction approach.

Procedure 6.2: Exact Order Reduction of an n-D F-M Model Using a Common Left Invariant subspace

Input : A given n-D F-M model $(\boldsymbol{A}, \boldsymbol{B}, C, D; r)$;
Output: A reduced-order n-D F-M model $(\hat{\boldsymbol{A}}, \hat{\boldsymbol{B}}, \hat{C}, \hat{D}; \hat{r})$;

1 *Step 1*: Find a common Left invariant subspace $\mathcal{W}$ of $A_1, \ldots, A_n$ with basis vectors $\boldsymbol{w}_1, \cdots, \boldsymbol{w}_{\tilde{r}}$ satisfying (6.33);

2 *Step 2*: Construct a nonsingular matrix T in the form of (6.41) and compute setting R in the form (6.42);

3 *Step 3*: Obtain a reduced-order F-M model $(\hat{\boldsymbol{A}}, \hat{\boldsymbol{B}}, \hat{C}, D; \hat{r})$ by

$$\hat{A}_i = T_2 A_i R_2, \tag{6.57a}$$

$$\hat{B}_i = T_2 B_k, \tag{6.57b}$$

$$\hat{C} = C R_2, \tag{6.57c}$$

with T_2 in (6.41) and R_2 in (6.42) ;

4 **return** $(\hat{\boldsymbol{A}}, \hat{\boldsymbol{B}}, \hat{C}, \hat{D}; \hat{r})$.

Example 6.4. *Consider the 2-D F-M model with order 4:*

$$A_1 = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & -1 & -2 & 2 \\ 1 & 2 & 3 & -3 \\ 0 & 1 & 3 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 1 & -1 & 0 & -2 \\ -1 & 2 & 1 & 2 \\ 0 & 1 & 2 & 1 \end{bmatrix},$$
$$B_1 = \begin{bmatrix} -1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} -1 \\ -2 \\ 2 \\ -1 \end{bmatrix}, \tag{6.58}$$
$$C = \begin{bmatrix} 1 & -1 & -2 & 0 \end{bmatrix}, \quad D = 0.$$

Step 1: For the matrices A_1, A_2, there exists a common left invariant subspace $\mathcal{V}$ with basis vectors

$$\boldsymbol{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \quad \boldsymbol{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{6.59}$$

such that $\boldsymbol{v}_l^{\mathrm{T}} B_i = \boldsymbol{0}$, $k = 1, 2$, $l = 1, 2$.

Step 2: Construct the nonsingular matrix

$$T = \begin{array}{c} \left[\begin{array}{cccc} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] \end{array} \begin{array}{l} \leftarrow \boldsymbol{v}_1^{\mathrm{T}} \\ \leftarrow \boldsymbol{v}_2^{\mathrm{T}} \\ \\ \\ \end{array}$$

$$= \begin{bmatrix} T_1 \\ T_2 \end{bmatrix}, \tag{6.60}$$

and compute

$$R = T^{-1} = \left[\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

$$= \begin{bmatrix} R_1 & R_2 \end{bmatrix}. \tag{6.61}$$

Step III: By (6.57a)*, one can obtain the lower-order F-M model:*

$$\begin{aligned} \hat{A}_1 &= \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}, \quad \hat{A}_2 = \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}, \\ \hat{B}_1 &= \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \quad \hat{B}_2 = \begin{bmatrix} -1 \\ -2 \end{bmatrix}, \\ \hat{C} &= \begin{bmatrix} 1 & 1 \end{bmatrix}, \quad D = 0. \end{aligned} \tag{6.62}$$

It is seen that the order of the new obtained F-M model (6.58) *is 2, which is lower than the order of 4 for the given one.*

Remark 6.3. *It can be confirmed that the 2-D F-M model* (6.58) *cannot be reduced by the common left eigenvector approach in Section 5.1.*

6.4 Reduction of the Roesser model with Common Invariant Subspace

In the previous section, a necessary and sufficient reducibility condition has been given for n-D F-M models based on common invariant subspace. In this section, we extend this result to n-D Roesser models.

Theorem 6.7. *For a given n-D Roesser model* $(A, B, C, D, \boldsymbol{r})$*, if*

i) *the matrices* $A_{\mathrm{c}1}, \ldots, A_{\mathrm{c}n}$ *have a common right invariant subspace* $\mathcal{V}$ *with a basis* $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{\tilde{r}}\}$ *satisfying*

$$C_{\mathrm{c}i}\boldsymbol{v}_k = \mathbf{0}, \quad i = 1, \ldots, n, \quad k = 1, \ldots, \tilde{r}, \tag{6.63a}$$

$$0 < \tilde{r}; \tag{6.63b}$$

ii) *or the matrices* $A_{\mathrm{r}1}, \ldots, A_{\mathrm{r}n}$ *have a common left invariant subspace* $\mathcal{W}$ *with a basis* $\{\boldsymbol{w}_1, \ldots, \boldsymbol{w}_{\tilde{r}}\}$ *satisfying*

$$\boldsymbol{w}^{\mathrm{T}}B_k = \mathbf{0}, \ k = 1, \ldots, n, \tag{6.64a}$$

$$0 < \hat{r}; \tag{6.64b}$$

then the given n-D Roesser model is reducible.

Proof. We provide only the proof for i). The proof for ii) is is similar.

Select any linearly independent vectors $\boldsymbol{\mu}_{i,1}, \ldots, \boldsymbol{\mu}_{i,\hat{r}_i} \in \mathbf{R}_i^n$, $i = 1, \ldots, n$, such that the matrix

$$T = \left[\begin{array}{ccc|ccc|c|ccc} \boldsymbol{\mu}_1 & \ldots & \boldsymbol{\mu}_{\tilde{r}} & \boldsymbol{\mu}_{1,1} & \cdots & \boldsymbol{\mu}_{1,\hat{r}_1} & \cdots & \boldsymbol{\mu}_{n,1} & \cdots & \boldsymbol{\mu}_{n,\hat{r}_n} \end{array}\right] = \left[\begin{array}{cc} \tilde{R} & R \end{array}\right] \tag{6.65}$$

is is nonsingular (invertible). Then, partition T^{-1} as

$$T^{-1} \triangleq \left[\begin{array}{c} \tilde{L} \\ L \end{array}\right] \tag{6.66}$$

with $\tilde{L} \in \mathbf{R}^{\tilde{r} \times r}$ and $L \in \mathbf{R}^{\hat{r} \times r}$.

Due to i) of Theorem 6.7, we have that $A_{\mathrm{c}i}\boldsymbol{\mu}_k$, $i = 1, \ldots, n$, $k = 1, \ldots, \tilde{r}$, can be expressed as a linear combination of the vectors $\boldsymbol{\mu}_1, \ldots, \boldsymbol{\mu}_{\tilde{r}}$. Thus,

$$\begin{aligned}
&A_{\mathrm{c}i}T \\
=&A_{\mathrm{c}i}\left[\begin{array}{ccc|ccc:c:ccc} \boldsymbol{\mu}_1 & \ldots & \boldsymbol{\mu}_{\tilde{r}} & \boldsymbol{\mu}_{1,1} & \cdots & \boldsymbol{\mu}_{1,\hat{r}_1} & \cdots & \boldsymbol{\mu}_{n,1} & \cdots & \boldsymbol{\mu}_{n,\hat{r}_n} \end{array}\right] \\
=&\left[\begin{array}{ccc|ccc:c:ccc} A_{\mathrm{c}i}\boldsymbol{\mu}_1 & \ldots & A_{\mathrm{c}i}\boldsymbol{\mu}_{\tilde{r}} & A_{\mathrm{c}i}\boldsymbol{\mu}_{1,1} & \cdots & A_{\mathrm{c}i}\boldsymbol{\mu}_{1,\hat{r}_1} & \cdots & A_{\mathrm{c}i}\boldsymbol{\mu}_{n,1} & \cdots & A_{\mathrm{c}i}\boldsymbol{\mu}_{n,\hat{r}_n} \end{array}\right] \\
=&\left[\begin{array}{ccc|ccc:c:ccc} \boldsymbol{\mu}_1 & \ldots & \boldsymbol{\mu}_{\tilde{r}} & \boldsymbol{\mu}_{1,1} & \cdots & \boldsymbol{\mu}_{1,\hat{r}_1} & \cdots & \boldsymbol{\mu}_{n,1} & \cdots & \boldsymbol{\mu}_{n,\hat{r}_n} \end{array}\right]\left[\begin{array}{c|c} \tilde{A}_{\mathrm{c}i} & \check{A}_{\mathrm{c}i} \\ \hline \mathbf{0} & \hat{A}_{\mathrm{c}i} \end{array}\right] \\
=&T\left[\begin{array}{c|c} \tilde{A}_{\mathrm{c}i} & \check{A}_{\mathrm{c}i} \\ \hline \mathbf{0} & \hat{A}_{\mathrm{c}i} \end{array}\right]
\end{aligned} \tag{6.67}$$

for some $\tilde{r} \times \tilde{r}$ matrix $\tilde{A}_{ci}$ and

$$\check{A}_{ci} = \tilde{L} A_{ci} R \in \mathbf{R}^{\tilde{r}\times\hat{r}},\ \hat{A}_{ci} = L A_{ci} R \in \mathbf{R}^{\hat{r}\times\hat{r}} \tag{6.68}$$

with $\hat{r} = \hat{r}_1 + \ldots + \hat{r}_n < r$. By the definitions of $\boldsymbol{\mu}_{i,k}$ and R, we have

$$ZR = \left[\hat{\boldsymbol{\mu}}_{1,1} z_1 \ldots \hat{\boldsymbol{\mu}}_{1,\hat{r}_1} z_1 \ldots \hat{\boldsymbol{\mu}}_{n,1} z_n \ldots \hat{\boldsymbol{\mu}}_{n,\hat{r}_n} z_n\right] = R\hat{Z} \tag{6.69}$$

with $\hat{Z} = \mathrm{diag}\{z_1 I_{\hat{r}_1}, \ldots, z_n I_{\hat{r}_n}\}$. In view of (6.67) and (6.69), we obtain

$$\begin{aligned} T^{-1}AZT &= T^{-1}\left(\sum_{i=1}^{n} A_{ci} z_i\right) T = \sum_{i=1}^{n} T^{-1} A_{ci} T z_i = \begin{bmatrix} \sum_{i=1}^{n} \tilde{A}_{ci} z_i & \sum_{i=1}^{n} \check{A}_{ci} z_i \\ \mathbf{0} & \sum_{i=1}^{n} \hat{A}_{ci} z_i \end{bmatrix} \\ &= \begin{bmatrix} \sum_{i=1}^{n} \tilde{A}_{ci} z_i & \tilde{L}AZR \\ \mathbf{0} & LAZR \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{n} \tilde{A}_{ci} z_i & \tilde{L}AR\hat{Z} \\ \mathbf{0} & LAR\hat{Z} \end{bmatrix}. \end{aligned} \tag{6.70}$$

It follows from (6.63a) that for every $i \in \{1, \ldots, n\}$,

$$\begin{aligned} C_{ci}T &= \left[\, C_{ci}\boldsymbol{\mu}_1 \ \ldots \ C_{ci}\boldsymbol{\mu}_{\tilde{r}} \,\middle|\, C_{ci}\boldsymbol{\mu}_{1,1} \ \cdots \ C_{ci}\boldsymbol{\mu}_{1,\hat{r}_1} \,\vdots \cdots \vdots\, C_{ci}\boldsymbol{\mu}_{n,1} \ \cdots \ C_{ci}\boldsymbol{\mu}_{n,\hat{r}_n} \,\right] \\ &= \left[\, \mathbf{0} \ \ldots \ \mathbf{0} \,\middle|\, C_{ci}\boldsymbol{\mu}_{1,1} \ \cdots \ C_{ci}\boldsymbol{\mu}_{1,\hat{r}_1} \ \cdots \ C_{ci}\boldsymbol{\mu}_{n,1} \ \cdots \ C_{ci}\boldsymbol{\mu}_{n,\hat{r}_n} \,\right] \\ &= \left[\, 0 \,\middle|\, \hat{C}_{ci} \,\right] \end{aligned}$$

with $\hat{C}_{ci} = C_{ci} R \in \mathbf{R}^{p\times\hat{r}}$. Thus,

$$CZT = \left(\sum_{i=1}^{n} C_{ci} z_i\right) T = \left[\, \mathbf{0} \ \ \sum_{i=1}^{n} C_{ci} z_i \,\right] = \left[\, \mathbf{0} \ \ CZR \,\right] = \left[\, \mathbf{0} \ \ CR\hat{Z} \,\right]. \tag{6.71}$$

We see from (6.70) and (6.71) that

$$\begin{aligned} &CZ(I_r - AZ)^{-1}B = CZTT^{-1}(I_r - AZ)^{-1}TT^{-1}B \\ &= (CZT)\left(I_r - T^{-1}AZT \right)^{-1} (T^{-1}B) \\ &= \left[\, \mathbf{0} \ \ CR\hat{Z} \,\right] \begin{bmatrix} I_2 - \sum_{i=1}^{n} \tilde{A}_{ci} z_i & -\tilde{L}AR\hat{Z} \\ \mathbf{0} & I_{\hat{r}} - LAR\hat{Z} \end{bmatrix}^{-1} \begin{bmatrix} \tilde{L}B \\ LB \end{bmatrix} \\ &= CR\hat{Z}\left(I_{\hat{r}} - LAR\hat{Z} \right)^{-1} LB. \end{aligned} \tag{6.72}$$

That is to say, we have obtained a new n-D Roesser model

$$(\hat{A}, \hat{B}, \hat{C}, D, \hat{\boldsymbol{r}}) \triangleq (LAR, LB, CR, D; \hat{\boldsymbol{r}})$$

with order $\hat{r} < r$. □

Theorem 6.8. *For a given n-D Roesser model $(A, B, C, D, \boldsymbol{r})$, if there is a low-order n-D $(\hat{A}, \hat{B}, \hat{C}, D, \hat{\boldsymbol{r}})$ such that they both have the same noncommutative transfer matrix, then*

i) *the matrices $A_{\mathrm{c}1}, \ldots, A_{\mathrm{c}n}$ have a common right invariant subspace $\mathcal{V}$ with a basis $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{\tilde{r}}\}$ satisfying*

$$C_{\mathrm{c}i}\boldsymbol{v}_k = \mathbf{0}, \quad i = 1, \ldots, n, \quad k = 1, \ldots, \tilde{r}, \tag{6.73a}$$

$$0 < \tilde{r}; \tag{6.73b}$$

ii) *or the matrices $A_{\mathrm{r}1}, \ldots, A_{\mathrm{r}n}$ have a common left invariant subspace $\mathcal{W}$ with a basis $\{\boldsymbol{w}_1, \ldots, \boldsymbol{w}_{\tilde{r}}\}$ satisfying*

$$\boldsymbol{w}^{\mathrm{T}}B_k = \mathbf{0}, \ k = 1, \ldots, n, \tag{6.74a}$$

$$0 < \hat{r}; \tag{6.74b}$$

then the given n-D Roesser model is reducible.

The proof is similar to the one for Theorem 6.6, and thus is omitted.

In what follows, a procedure to obtain a reduced n-D Roesser model is given based on common right invariant subspace.

Procedure 6.3: Exact Order Reduction of an n-D Roesser Model Using a Common Rigth Invariant subspace

---

Input : A given n-D Roesser model $(A, B, C, D; \boldsymbol{r})$;
Output: A reduced Roesser model $(\hat{A}, \hat{B}, \hat{C}, D; \hat{\boldsymbol{r}})$;

1 *Step 1*: Select vectors $\boldsymbol{\mu}_{i,1}, \ldots, \boldsymbol{\mu}_{i,\hat{r}_i} \in \mathbf{R}_i^n$ to construct a nonsingular matrix T in the form of (6.65);

2 *Step 2*: Extract R and L from T of (6.65) and T^{-1} of (6.66), respectivley;

3 *Step 3*: Obtain a reduced Roesser model $(\hat{A}, \hat{B}, \hat{C}, D; \hat{\boldsymbol{r}})$:

$$\hat{A} \triangleq LAR, \ \hat{B} \triangleq LB, \ \hat{C} \triangleq CR; \tag{6.75}$$

4 **return** $(\hat{A}, \hat{B}, \hat{C}, D; \hat{\boldsymbol{r}}) \triangleq (A, B, C, D; \boldsymbol{r})$,

---

Remark 6.4. *Due to the duality of the common right common invariant subspace and common left common invariant subspace, the procedure to obtain a reduced n-D Roesser* *model based on a common left common invariant subsapce can be done as follows. For a given n-D Roesser model* $(A, B, C, D; \boldsymbol{r})$ *if the matrices* $A_{\mathrm{r}1}, \ldots, A_{\mathrm{r}n}$ *have a common left invariant subspace* $\mathcal{W}$ *with a basis*

$$\{\boldsymbol{\omega}_1, \ldots, \boldsymbol{\omega}_{\hat{r}}\} \tag{6.76}$$

such that (6.63) *holds true, the matrices* $A_{\mathrm{r}1}^{\mathrm{T}}, \ldots, A_{\mathrm{r}n}^{\mathrm{T}}$ *have a common right invariant subspace* $\mathcal{W}$ *with a basis in* (6.76) *such that*

$$B_{\mathrm{r}k}^{\mathrm{T}}\boldsymbol{\omega} = \mathbf{0}; \tag{6.77a}$$

$$0 < \hat{r}. \tag{6.77b}$$

This shows that the n-D Roesser model $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}; \tilde{\boldsymbol{r}}) \triangleq (A^{\mathrm{T}}, B^{\mathrm{T}}, C^{\mathrm{T}}, D^{\mathrm{T}}; \boldsymbol{r})$ *can be reduced. Then, by applying Procedure 5.2 to the n-D Roesser model* $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}; \tilde{\boldsymbol{r}})$*, one can obtain a reduced Roesser model* $(\hat{\tilde{A}}, \hat{\tilde{B}}, \hat{\tilde{C}}, \hat{\tilde{D}}; \tilde{\boldsymbol{r}})$*. Finally, one can obtain a reduced Roesser model* $(\hat{A}, \hat{B}, \hat{C}, D, \hat{\boldsymbol{r}}) \triangleq (\hat{\tilde{A}}^{\mathrm{T}}, \hat{\tilde{B}}^{\mathrm{T}}, \hat{\tilde{C}}^{\mathrm{T}}, \hat{\tilde{D}}^{\mathrm{T}}; \hat{\boldsymbol{r}})$ *for the given Roesser model* $(A, B, C, D; \boldsymbol{r})$*.*

In view of Theorems 6.4 and 6.8, one can obtain the following result.

Theorem 6.9. *For a given n-D Roesser model* $(A, B, C, D; \boldsymbol{r})$*, and suppose* $\mathcal{R}_r(\boldsymbol{A}_{\mathrm{r}}, \boldsymbol{B}_{\mathrm{r}})$ *and* $\mathcal{O}_r(\boldsymbol{A}_{\mathrm{c}}, \boldsymbol{C}_{\mathrm{c}})$ *are associated r-step reachability matrix and observability matrix, respectively. Then, the n-D Roesser model* $(A, B, C, D; \boldsymbol{r})$ *is a minimal noncommutative Roesser model if and only if* $\mathrm{rank}(\mathcal{R}_r)(\boldsymbol{A}_{\mathrm{R}}, \boldsymbol{A}_{\mathrm{R}}) = r$ *and* $\mathrm{rank}\left(\mathcal{O}_r(\boldsymbol{A}_{\mathrm{C}}, \boldsymbol{C}_{\mathrm{C}})\right) = r$*.*

Next, it will be shown that the minimal noncommutative Roesser model are tightly connected. To this end, we need the notion of structured similarity transformation matrix.

Definition 6.8. *[31, 34] The structured similarity transformation matrix T for an n-D Roesser state-space model* $(A, B, C, D; \boldsymbol{r})$ *is defined in the form of*

$$T \triangleq \mathrm{diag}\{T_{1,1}, \ldots, T_{n,n}\} \tag{6.78}$$

where each $T_{i,i} \in \mathbf{C}^{r_i \times r_i}$ *is nonsingular.*

Given an n-D Roesser state-space model $(A, B, C, D; \boldsymbol{r})$ and any T in the form of (6.78), it can be verified that

$$(CTZ)\left(I - T^{-1}ATZ\right)^{-1}(T^{-1}B) + D = (CZ)(I - AZ)^{-1}(B) + D. \tag{6.79}$$

Therefore, for any n-D Roesser model $(A, B, C, D; \boldsymbol{r})$ and any T in the form of (6.78), an e-quivalent n-D Roesser model can be obtained by $(\tilde{A}, \tilde{B}, \tilde{C}, D; \tilde{\boldsymbol{r}}) \triangleq (T^{-1}AT, T^{-1}B, CT, D; \boldsymbol{r})$. The n-D Roesser state-space model $(\tilde{A}, \tilde{B}, \tilde{C}, D; \tilde{\boldsymbol{r}})$ is said to be obtained by a structure similarity transformation from the n-D Roesser model $(A, B, C, D; \boldsymbol{r})$. Then, we have the following results.

We have the following results.

Lemma 6.2. *Two n-D Roesser state-space models $(A, B, C, D; r)$ and $(\tilde{A}, \tilde{B}, \tilde{C}, D; \tilde{\boldsymbol{r}})$ are minimal noncommutative realizations of a same noncommutative transfer matrix, i.e.,*

$$(\tilde{C}_w \tilde{A}_w \tilde{B}_w) = (C_w A_w B_w) \tag{6.80}$$

for any $w = i_1 i_2 \dots i_l \in \mathcal{F}_n^+$, if and only if these two n-D Roesser models are related by

$$(\tilde{A}, \tilde{B}, \tilde{C}, D; \tilde{\boldsymbol{r}}) = (T^{-1}AT, T^{-1}B, CT, D; \boldsymbol{r}) \tag{6.81}$$

where T is any structured similarity transformation matrix for the n-D Roesser state-space model $(A, B, C, D; r)$.

Proof. The proof of the necessary part is similar to that of the necessary part for Theorem 8.10 on page 320 in [88] and thus is omitted here. We only give the proof of the sufficient part. Note that

$$\tilde{A} = T^{-1}AT = \begin{bmatrix} T_{1,1}^{-1}A_{1,1}T_{1,1} & \dots & T_{1,1}^{-1}A_{1,n}T_{n,n} \\ \vdots & & \vdots \\ T_{n,n}^{-n}A_{n,1}T_{1,1} & \dots & T_{n,n}^{-1}A_{n,n}T_{n,n} \end{bmatrix}, \tag{6.82a}$$

$$\tilde{B} = T^{-1}B = \begin{bmatrix} T_{1,1}^{-1}B_1 \\ \vdots \\ T_{n,n}^{-1}B_n \end{bmatrix}, \tag{6.82b}$$

$$C = CT = \begin{bmatrix} C_1T_{1,1} & \dots & C_nT_{n,n} \end{bmatrix}. \tag{6.82c}$$

Then, for any $w = i_1 i_2 \dots i_l$, we have

$$\begin{aligned} &(\tilde{C}_w \tilde{A}_w \tilde{B}_w) \\ =&(C_{i_l}T_{i_l,i_l})\big((T_{i_l,i_l}^{-1}A_{i_l,i_{l-1}}T_{i_{l-1},i_{l-1}}) \dots (T_{i_2,i_2}^{-1}A_{i_2,i_1}T_{i_1,i_1})\big)(T_{i_1,i_1}^{-1}B_{i_1}) \\ =&C_{i_l}(A_{i_l,i_{l-1}}A_{i_{l-1},i_{l-2}} \dots A_{i_2,i_1})B_{i_1} = (C_w A_w B_w) \end{aligned} \tag{6.83}$$

□

Lemma 6.3. *If an n-D Roesser state-space model* $(\tilde{A}, \tilde{B}, \tilde{C}, D; \tilde{\boldsymbol{r}})$ *can be obtained in the form of*

$$\begin{aligned}
&\tilde{A} \triangleq T^{-1}AT = \begin{bmatrix} \tilde{A}_{1,1} & \dots & \tilde{A}_{1,n} \\ \vdots & \ddots & \vdots \\ \tilde{A}_{n,1} & \dots & \tilde{A}_{n,n} \end{bmatrix}, \quad \tilde{B} \triangleq T^{-1}B = \begin{bmatrix} \tilde{B}_1 \\ \vdots \\ \tilde{B}_n \end{bmatrix}, \\
&\tilde{C} \triangleq CT = \begin{bmatrix} \tilde{C}_1 & \cdots & \tilde{C}_n \end{bmatrix},
\end{aligned} \tag{6.84}$$

with

$$\tilde{A}_{i,j} = \begin{bmatrix} \hat{A}_{i,j} & \tilde{A}_{i,j,2} \\ \mathbf{0} & \tilde{A}_{i,j,4} \end{bmatrix}, \quad \tilde{B}_i = \begin{bmatrix} \hat{B}_i \\ \mathbf{0} \end{bmatrix}, \quad \tilde{C}_k = \begin{bmatrix} (\hat{C}_k) & (\tilde{C}_{k,2}) \end{bmatrix}. \tag{6.85}$$

and $i, k = 1, \dots, n$ *from a given n-D Roesser state-space model* $(A, B, C, D; \boldsymbol{r})$ *by appropriate structured similarity transformations. Then, the two Roesser models* $(\hat{A}, \hat{B}, \hat{C}, D; \hat{\boldsymbol{r}})$ *and* $(A, B, C, D; \boldsymbol{r})$ *satisfy the relation (6.11a), where*

$$\hat{A} = \begin{bmatrix} \hat{A}_{1,1} & \dots & \hat{A}_{1,n} \\ \vdots & \ddots & \vdots \\ \hat{A}_{n,1} & \dots & \hat{A}_{n,n} \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} \hat{B}_1 \\ \vdots \\ \hat{B}_n \end{bmatrix}, \quad \hat{C} = \begin{bmatrix} \hat{C}_1 & \dots & \hat{C}_n \end{bmatrix}. \tag{6.86}$$

Proof. Noting that structured similar transformations do not change the noncommutative transfer matrix of systems, we than have for any $w = \alpha_1\alpha_2 \dots \alpha_l \in \mathcal{F}_{\mathbf{n}}^+$,

$$\begin{aligned}
C_{\alpha_l} A_w B_{\alpha_1} &= \tilde{C}_{\alpha_l} \tilde{A}_w \tilde{B}_{\alpha_1} = \tilde{C}_{\alpha_l} \left(\tilde{A}_{\alpha_l,\alpha_{l-1}} \dots \tilde{A}_{\alpha_2,\alpha_1} \right) \tilde{B}_{\alpha_1} \\
&= [\tilde{C}^1_{\alpha_l} \; \tilde{C}^2_{\alpha_l}] \left(\begin{bmatrix} \tilde{A}^{1,1}_{\alpha_l,\alpha_{l-1}} & \tilde{A}^{1,2}_{\alpha_l,\alpha_{l-1}} \\ \mathbf{0} & \tilde{A}^{2,2}_{\alpha_l,\alpha_{l-1}} \end{bmatrix} \dots \begin{bmatrix} \tilde{A}^{1,1}_{\alpha_2,\alpha_1} & \tilde{A}^{1,2}_{\alpha_2,\alpha_1} \\ \mathbf{0} & \tilde{A}^{2,2}_{\alpha_2,\alpha_1} \end{bmatrix} \right) \begin{bmatrix} \tilde{B}^1_{\alpha_1} \\ \mathbf{0} \end{bmatrix} \\
&= \hat{C}_{\alpha_l} \left(\hat{A}_{\alpha_l,\alpha_{l-1}} \dots \hat{A}_{\alpha_2,\alpha_1} \right) \hat{B}_{\alpha_1},
\end{aligned} \tag{6.87}$$

with $w = \alpha_1\alpha_2 \dots \alpha_l \in \mathcal{F}_{\mathbf{n}}^+$, A_{*v} defined in (6.3) and z^w defined in (6.5). Noting also $\hat{r}_1 + \dots \hat{r}_n < \tilde{r}_1 + \dots \tilde{r}_n = r_1 + \dots r_n$, and thus $\hat{r} < r$. □

6.5 Contribution Summary

Necessary and sufficient conditions have been established for existence of common invariant subspace of multiple matrices. Based on these results, new necessary and sufficient conditions for the minimal state-space models realization of n-D systems in the sense of non-commutation have been developed. Moreover, basic procedures have been presented for exactly reducing the n-D F-M mode and the Roesser model, respectively. It has been shown that the new approach can be applied even to those systems for which the existing approach based on eigenvalues cannot do any further exact order reduction on them. Examples have been given to illustrate the details and effectiveness of the proposed the common invariant subspace.

Chapter 7

Further Exact Order Reduction

In the proceeding chapter, it has been shown that one can exactly reduce a given n-D Roesser model to a minimal noncommutative realization based on common invariant subspace. However, an general n-D Roesser model is a minimal noncommutative Roesser model does not mean that it is a minimal general n-D Roesser model. This chapter is to further study the exact order reduction of n-D Roesser model based on equivalence. Specifically, two types of transformations to obtain equivalent realizations are established for the Roesser model. It turns out that applying the these transformations can convert a given minimal n-D Roesser model in noncommutative setting to another Rosser model with different noncommutative transfer matrices and then the newly obtained Roesser model may be reduced again by applying the common invariant subspace approach. Based on this fact, a novel reduction procedure is presented, which repeatedly applies the common invariant subspace approach to generate minimal Roesser model realization in the noncommutative setting and the two equivalent transformations to obtain another Roesser model with different noncommutative transfer function matrices, such that an n-D Roesser model with order as low as possible can be obtained.

This Chapter is organized as follows. Section 7.1 provides motivation on our further study. In Section 7.2, a new basic reduce procedure to further exactly reduce Roesser models is presented by using two types of transformations: non-structured similar transformation and general transformation. Section 7.3 describes the similar transformation, while the general transformation is given in 7.4. Some main proofs are given in Section 7.6. Finally, conclusions are given in section 7.7.

7.1 Motivation

To intuitively expose that the minimality in the noncommutative case does not mean the minimality in the general commutative Roesser model, let us consider the following simple example.

Example 7.1. *Consider the 2-D Roesser state-space model* $(\bar{A}, \bar{B}, \bar{C}, D; \bar{\boldsymbol{r}})$*:*

$$\begin{aligned}&\bar{A} = \begin{bmatrix} \bar{A}_{1,1} & \bar{A}_{1,2} \\ \bar{A}_{2,1} & \bar{A}_{2,2} \end{bmatrix} = \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & 1 \\ \hline 0 & 1 & 2 \end{array}\right], \quad \bar{B} = \begin{bmatrix} \bar{B}_1 \\ \bar{B}_2 \end{bmatrix} = \left[\begin{array}{c} 1 \\ 0 \\ \hline 1 \end{array}\right], \\ &\bar{C} = \begin{bmatrix} \bar{C}_1 & \bar{C}_2 \end{bmatrix} = \left[\begin{array}{cc|c} 1 & 1 & 0 \end{array}\right], \quad D = 0,\end{aligned} \tag{7.1}$$

which is the realizations of

$$H(z_1, z_2) = \frac{-z_1 z_2 + z_1}{z_1 z_2 - z_1 - 2z_2 + 1}. \tag{7.2}$$

For the 2-D Roesser state-space model $(\bar{A}, \bar{B}, \bar{C}, D; \bar{\boldsymbol{r}})$*, we have*

$$\begin{aligned}&\bar{A}_{c1} = \begin{bmatrix} \bar{A}_{1,1} & \mathbf{0}_{2,1} \\ \bar{A}_{2,1} & \mathbf{0}_{1,1} \end{bmatrix} = \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 1 & 0 \end{array}\right], \quad \bar{A}_{c2} = \begin{bmatrix} \mathbf{0}_{2,2} & \bar{A}_{1,2} \\ \mathbf{0}_{1,2} & \bar{A}_{2,2} \end{bmatrix} = \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 1 \\ \hline 0 & 0 & 2 \end{array}\right], \\ &\bar{C}_{c1} = \begin{bmatrix} \bar{C}_1 & \mathbf{0}_{1,1} \end{bmatrix} = \left[\begin{array}{cc|c} 1 & 1 & 0 \end{array}\right], \quad \bar{C}_{c2} = \begin{bmatrix} \mathbf{0}_{1,2} & \bar{C}_2 \end{bmatrix} = \left[\begin{array}{cc|c} 0 & 0 & 0 \end{array}\right],\end{aligned} \tag{7.3}$$

and

$$\begin{aligned}&\bar{A}_{r1} = \begin{bmatrix} \bar{A}_{1,1} & \bar{A}_{1,2} \\ \bar{A}_{2,1} & \bar{A}_{2,2} \end{bmatrix} = \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & 1 \\ \hline 0 & 0 & 0 \end{array}\right], \quad \bar{A}_{r2} = \begin{bmatrix} \bar{A}_{1,1} & \bar{A}_{1,2} \\ \mathbf{0}_{1,2} & \mathbf{0}_{1,1} \end{bmatrix} = \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 1 & 2 \end{array}\right], \\ &\bar{B}_{r1} = \begin{bmatrix} \bar{B}_1 \\ \mathbf{0}_{1,1} \end{bmatrix} = \left[\begin{array}{c} 1 \\ 0 \\ \hline 0 \end{array}\right], \quad \bar{B}_{r2} = \begin{bmatrix} \mathbf{0}_{2,1} \\ \bar{B}_2 \end{bmatrix} = \left[\begin{array}{c} 0 \\ 0 \\ \hline 1 \end{array}\right],\end{aligned} \tag{7.4}$$

One can compute

$$\mathcal{O}_{3,1}(\bar{\boldsymbol{A}}_c, \bar{\boldsymbol{C}}_c) = \begin{bmatrix} \bar{C}_{c1} \\ \bar{C}_{c1}\bar{A}_{c1} \\ \bar{C}_{c2}\bar{A}_{c1} \\ \bar{C}_{c1}\bar{A}_{c1}\bar{A}_{c1} \\ \bar{C}_{c2}\bar{A}_{c1}\bar{A}_{c1} \\ \bar{C}_{c1}\bar{A}_{c2}\bar{A}_{c1} \\ \bar{C}_{c2}\bar{A}_{c2}\bar{A}_{c1} \end{bmatrix} = \left[\begin{array}{cc|c} 1 & 1 & 0 \\ \hline 1 & 2 & 0 \\ \hline 0 & 0 & 0 \\ \hline 1 & 3 & 0 \\ \hline 0 & 0 & 0 \\ \hline 0 & 1 & 0 \\ \hline 0 & 0 & 0 \end{array}\right], \quad \mathcal{O}_{3,2}(\bar{\boldsymbol{A}}_c, \bar{\boldsymbol{C}}_c) = \begin{bmatrix} \bar{C}_{c2} \\ \bar{C}_{c1}\bar{A}_{c2} \\ \bar{C}_{c2}\bar{A}_{c2} \\ \bar{C}_{c1}\bar{A}_{c1}\bar{A}_{c2} \\ \bar{C}_{c2}\bar{A}_{c1}\bar{A}_{c2} \\ \bar{C}_{c1}\bar{A}_{c2}\bar{A}_{c2} \\ \bar{C}_{c2}\bar{A}_{c2}\bar{A}_{c2} \end{bmatrix} = \left[\begin{array}{cc|c} 0 & 0 & 0 \\ \hline 0 & 0 & 1 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & 2 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & 2 \\ \hline 0 & 0 & 0 \end{array}\right]. \tag{7.5}$$

and

$$\begin{aligned}
\mathcal{R}_{3,1}(\bar{\boldsymbol{A}}_{\rm r};\bar{\boldsymbol{B}}_{\rm r}) &= \left[\begin{array}{ccccccc} \bar{B}_{\rm r1} & \bar{A}_{\rm r1}\bar{B}_{\rm r1} & \bar{A}_{\rm r1}\bar{B}_{\rm r2} & \bar{A}_{\rm r1}\bar{A}_{\rm r1}\bar{B}_{\rm r1} & \bar{A}_{\rm r1}\bar{A}_{\rm r1}\bar{B}_{\rm r2} & \bar{A}_{\rm r1}\bar{A}_{\rm r2}\bar{B}_{\rm r1} & \bar{A}_{\rm r1}\bar{A}_{\rm r2}\bar{B}_{\rm r2} \end{array}\right] \\
&= \left[\begin{array}{c|c|c|c|c|c|c} 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right], \\
\mathcal{R}_{3,2}(\bar{\boldsymbol{A}}_{\rm r};\bar{\boldsymbol{B}}_{\rm r}) &= \left[\begin{array}{ccccccc} \bar{B}_{\rm r2} & \bar{A}_{\rm r2}\bar{B}_{\rm r1} & \bar{A}_{\rm r2}\bar{B}_{\rm r2} & \bar{A}_{\rm r2}\bar{A}_{\rm r1}\bar{B}_{\rm r1} & \bar{A}_{\rm r2}\bar{A}_{\rm r1}\bar{B}_{\rm r2} & \bar{A}_{\rm r2}\bar{A}_{\rm r2}\bar{B}_{\rm r1} & \bar{A}_{\rm r2}\bar{A}_{\rm r2}\bar{B}_{\rm r2} \end{array}\right] \\
& \left[\begin{array}{c|c|c|c|c|c|c} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & 1 & 0 & 4 \end{array}\right].
\end{aligned} \tag{7.6}$$

One can verify that

$$\operatorname{rank}\left(\mathcal{R}_{3,i}(\bar{\boldsymbol{A}}_{\rm r};\bar{\boldsymbol{B}}_{\rm r})\right) = \operatorname{rank}\left(\mathcal{O}_{3,i}(\bar{\boldsymbol{A}}_{\rm c};\bar{\boldsymbol{C}}_{\rm c})\right) = \hat{r}_i, \quad i = 1, 2. \tag{7.7}$$

Therefore, by Theorem 6.9 we have that the 2*-D Roesser model* $(\bar{A}, \bar{B}, \bar{C}, D; \bar{\boldsymbol{r}})$ *in* (7.1) *is a minimal noncommutative Roesser model and cannot be further reduced by the common invariant subspace approach.*

However, there is a lower-order 2-D Roesser state-space model $(\check{A}, \check{B}, \check{C}, D; \check{r})$

$$\begin{aligned}
\check{A} &= \begin{bmatrix} \check{A}_{1,1} & \check{A}_{1,2} \\ \check{A}_{2,1} & \check{A}_{2,2} \end{bmatrix} = \left[\begin{array}{c|c} 1 & 1 \\ \hline 1 & 2 \end{array}\right], \quad \check{B} = \begin{bmatrix} \check{B}_1 \\ \check{B}_2 \end{bmatrix} = \left[\begin{array}{c} 1 \\ \hline 0 \end{array}\right], \\
\check{C} &= \begin{bmatrix} \check{C}_1 & \check{C}_2 \end{bmatrix} = \left[\begin{array}{c|c} 1 & 1 \end{array}\right], \quad D = 0,
\end{aligned} \tag{7.8}$$

having the same transfer function with that of the 2-D Roesser state-space model $(\bar{A}, \bar{B}, \bar{C}, D; \bar{r})$ in (7.1). Therefore the minimal noncommutative realization does not mean the minimal realization in the commutative setting.

Moreover, it is well known that the minimal state-space realization of n-D systems is an extremely difficult problem, and, for a general n-D system, we even have neither any (necessary and sufficient) condition for the existence of an absolutely minimal realization nor any condition to test whether or not a non-absolutely-minimal realization is minimal (or relatively minimal) [11, 32, 51].

The motivation of our research is to furtehr study the exact order reduction of the n-D Roesser models by equivalent transformation, which can transform one minimal noncommutative Roesser model to other Roesser model with different noncommutative transfer function matrices but with the same transfer matrix in commutative setting.

7.2 Equivalent Realizations

In this section, two types of transformations to obtain equivalent realizations for n-D Roesser models will be presented to convert an n-D Roesser model to another one such that they both have different noncommutative transfer function matrices but have the same transfer matrix in commutative setting. With these transformations, a new exact order reduction approach for n-D Roesser models will be proposed and a basic exact order reduction procedure will be given.

For n-D Roesser models, the equivalence is defined as follows.

Definition 7.1. *Two realizations $(\bar{A}, \bar{B}, \bar{C}, D; \bar{\boldsymbol{r}})$ and $(A, B, C, D; \boldsymbol{r})$ are equivalent if*

$$\bar{C}\bar{Z}(I_{\bar{r}} - \bar{A}\bar{Z})^{-1}\bar{B} + D = CZ(I_r - AZ)^{-1}B + D. \tag{7.9}$$

Note that the n-D Roesser state-space model $(\bar{A}, \bar{B}, \bar{C}, D; \boldsymbol{r})$ and $(A, B, C, D; \boldsymbol{r})$ may be with possibly different dimensions, i.e., $\bar{r}_i \neq r_i$ for some $i \in \{1, \ldots, n\}$.

7.2.1 Non-Structured Similarity Transformations

The non-structured similarity transformations are defined as follows.

Definition 7.2. *For an n-D Roesser state-space model $(\bar{A}, \bar{B}, \bar{C}, D; \bar{\boldsymbol{r}})$, the non-structured similarity transformation matrix P is a nonsingular matrix*

$$P \triangleq \begin{bmatrix} P_{1,1} & \cdots & P_{1,n} \\ \vdots & \ldots & \vdots \\ P_{n,1} & \cdots & P_{n,n} \end{bmatrix}, \tag{7.10}$$

with $P_{i,j} \in \mathbf{R}^{r_i \times r_j}$ is not all zero matrix for $i \neq j$ and $i, j \in \bar{n}$ such that the n-D Roesser models $(A, B, C, D; \boldsymbol{r})$ and $(\bar{A}, \bar{B}, \bar{C}, D; \bar{\boldsymbol{r}})$ have the following relation

$$CZ(I - AZ)B + D = \bar{C}(I_{\bar{r}} - \bar{A}\bar{Z})\bar{B} + D, \tag{7.11}$$

where $(A, B, C, D; \boldsymbol{r}) \triangleq (P^{-1}\bar{A}P, P^{-1}\bar{B}, \bar{C}P, D; \bar{\boldsymbol{r}})$. The n-D Roesser model $(A, B, C, D; \boldsymbol{r})$ is said to be obtained by a non-structure similarity transformation from the n-D Roesser model $(\bar{A}, \bar{B}, \bar{C}, D; \bar{\boldsymbol{r}})$.

The following example is given to show that an equivalent realization can be obtained from a given n-D Roesser model by a non-structure similarity transformation matrix.

Example 7.2. *Consider the minimal noncommutative 2-D Roesser model realization* $(\bar{A}, \bar{B}, \bar{C}, D; (2,1))$ *in* (7.1). *Using a non-structure similarity transformation matrix*

$$P = \begin{bmatrix} P_{1,1} & P_{1,2} \\ P_{2,1} & P_{2,2} \end{bmatrix} = \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 1 \\ \hline 0 & 1 & 1 \end{array}\right], \tag{7.12}$$

one can obtain a new n-D Roesser model $(A, B, C, D; \boldsymbol{r}) \triangleq (P^{-1}\bar{A}P, P^{-1}\bar{B}, \bar{C}P, D; \bar{\boldsymbol{r}})$*:*

$$\begin{aligned} A &= \begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{bmatrix} = \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \\ \hline 0 & 1 & 2 \end{array}\right], \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \left[\begin{array}{c} 1 \\ 1 \\ \hline 0 \end{array}\right], \\ C &= \begin{bmatrix} C_1 & C_2 \end{bmatrix} = \left[\begin{array}{cc|c} 1 & 0 & 1 \end{array}\right], \quad D = 0, \end{aligned} \tag{7.13}$$

such that the n-D Roesser state-space models $(A, B, C, D; \boldsymbol{r})$ *and* $(\bar{A}, \bar{B}, \bar{C}, D; \bar{\boldsymbol{r}})$ *have the same transfer matrix* $H(z_1, z_2)$ *in (7.2). Note that* $P_{2,1} = \begin{bmatrix} 0 & 1 \end{bmatrix}$ *of* P *in (7.12) is not a zero matrix, then the matrix* P *is a non-structured similarity transformation matrix for the* 2*-D Roesser state-space model* $(\bar{A}, \bar{B}, \bar{C}, D; \bar{\boldsymbol{r}})$ *in (7.1). And the* 2*-D Roesser state-space model* $(A, B, C, D; \boldsymbol{r})$ *in (7.13) is said to be obtained by a non-structure similarity transformation matrix* P *in (7.12) from the* 2*-D Roesser model* $(\bar{A}, \bar{B}, \bar{C}, D; \bar{\boldsymbol{r}})$ *in (7.1).*

It should be noted that the relationship in (7.11) may not hold for an n-D Roesser model $(A, B, C, D; \boldsymbol{r}) \triangleq (T^{-1}\bar{A}T, T^{-1}\bar{B}, \bar{C}T, D; \bar{\boldsymbol{r}})$ obtained by any nonsingular matrix P from any n-D Roesser model $(\bar{A}, \bar{B}, \bar{C}, D; \bar{\boldsymbol{r}})$. To intuitively show this fact, let us consider the following simple example.

Example 7.3. *Consider again the minimal noncommutative 2-D Roesser model realization* $(\bar{A}, \bar{B}, \bar{C}, D; \bar{\boldsymbol{r}})$ *in (7.1) of* $H(z_1, z_2)$ *in (7.2) and choose a nonsingular matrix*

$$P = \begin{bmatrix} P_{1,1} & P_{1,2} \\ P_{2,1} & P_{2,2} \end{bmatrix} = \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 0 & 1 \\ \hline 0 & 1 & 1 \end{array}\right]. \tag{7.14}$$

Now, by the nonsingular matrix P *in (7.14), one can get a new* 2*-D Roesser model* $(A, B, C, D; \boldsymbol{r}) \triangleq (P^{-1}\bar{A}P, P^{-1}\bar{B}, \bar{C}P, D; \bar{\boldsymbol{r}})$*:*

$$A = \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 1 & 1 \\ \hline 0 & 1 & 2 \end{array}\right], \quad B = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad C = \left[\begin{array}{cc|c} 1 & 0 & 2 \end{array}\right], \quad D = 0, \tag{7.15}$$

whose transfer matrix,

$$H(z_1, z_2) = \frac{-{z_1}^2 z_2 + 2{z_1}^2 - z_1}{{z_1}^2 z_2 - {z_1}^2 - 3z_1 z_2 + 2z_1 + 2z_2 - 1}, \tag{7.16}$$

is not equal to $H(z_1, z_2)$ *in (7.2).*

From Example 7.3, it can be seen that not all nonsingular matrices are able to be used in the non-structured similarity transformations for any given n-D Roesser models. In Section 7.3, conditions and the corresponding procedure will be developed that an n-D Roesser model can be transformed to another equivalent realization by a nonsingular matrix.

7.2.2 General Transformations

In structured similarity transformations and non-structured similarity transformations, it is seen that an equivalent realization may be obtained by a nonsingular matrix T, i.e, structured similarity matrix or non-structured similarity matrix. However, there are some equivalent n-D Roesser state-space models in which one cannot be obtained from others by a nonsingular transformation. To intuitively show this fact, let us consider the following simple example.

Example 7.4. *Consider the minimal noncommutative 2-D Roesser state-space model* $(A, B, C, D; (3, 2))$*:*

$$\begin{aligned} \bar{A} &= \left[\begin{array}{ccc|cc} 2 & 0 & 0 & 0 & -2 \\ 0 & 1 & 2 & 1 & 2 \\ 0 & 0 & 2 & 0 & 1 \\ \hline 1 & -1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & -\frac{3}{2} \end{array}\right], \quad \bar{B} = \left[\begin{array}{cc} 2 & 2 \\ -6 & -4 \\ 0 & 0 \\ \hline 0 & 1 \\ 1 & 1 \end{array}\right], \\ \bar{C} &= \left[\begin{array}{ccc|cc} -1 & 2 & -2 & -1 & 0 \\ 1 & 0 & -2 & -1 & 0 \end{array}\right], \quad D = \left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array}\right], \end{aligned} \tag{7.17}$$

and the 2-D Roesser state-space model $(A, B, C, D; (2, 3))$*:*

$$\begin{aligned} A &= \left[\begin{array}{cc|ccc} 1 & 2 & 0 & 1 & 4 \\ 0 & 2 & -\frac{1}{2} & 0 & 2 \\ \hline 0 & 0 & -\frac{1}{2} & 0 & -1 \\ -1 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -\frac{3}{2} \end{array}\right], \quad B = \left[\begin{array}{cc} -8 & -6 \\ -1 & -1 \\ \hline 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{array}\right], \\ C &= \left[\begin{array}{cc|ccc} 2 & -2 & 0 & -1 & 0 \\ 0 & -2 & 0 & -1 & 0 \end{array}\right], \quad D = \left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array}\right], \end{aligned} \tag{7.18}$$

with the same total order and different partial orders, which are all the realizations of the same transfer matrix

$$H(z_1, z_2) = \begin{bmatrix} h_{1,1} & h_{1,2} \\ h_{2,1} & h_{2,2} \end{bmatrix}, \tag{7.19}$$

with

$$\begin{aligned}
h_{1,1} =& \frac{96{z_1}^2{z_2}^2 + 154{z_1}^2 z_2 + 52{z_1}^2 - 48 z_1 {z_2}^2 - 78 z_1 z_2 - 28 z_1}{6{z_1}^2 z_2 + 4{z_1}^2 - 6 z_1 {z_2}^2 - 13 z_1 z_2 - 6 z_1 + 3{z_2}^2 + 5 z_2 + 2} \\
h_{1,2} =& \frac{42{z_1}^2{z_2}^2 + 94{z_1}^2 z_2 + 36{z_1}^2 - 15 z_1 {z_2}^2 - 44 z_1 z_2 - 20 z_1 - 3{z_2}^2 - 2 z_2}{6{z_1}^2 z_2 + 4{z_1}^2 - 6 z_1 {z_2}^2 - 13 z_1 z_2 - 6 z_1 + 3{z_2}^2 + 5 z_2 + 2} \\
h_{2,1} =& \frac{32{z_1}^2{z_2}^2 + 30{z_1}^2 z_2 - 4{z_1}^2 - 16 z_1 {z_2}^2 - 14 z_1 z_2 + 4 z_1}{6{z_1}^2 z_2 + 4{z_1}^2 - 6 z_1 {z_2}^2 - 13 z_1 z_2 - 6 z_1 + 3{z_2}^2 + 5 z_2 + 2} \\
h_{2,2} =& \frac{14{z_1}^2{z_2}^2 + 18{z_1}^2 z_2 - 4{z_1}^2 - z_1 {z_2}^2 - 4 z_1 z_2 + 4 z_1 - 3{z_2}^2 - 2 z_2}{6{z_1}^2 z_2 + 4{z_1}^2 - 6 z_1 {z_2}^2 - 13 z_1 z_2 - 6 z_1 + 3{z_2}^2 + 5 z_2 + 2}
\end{aligned} \tag{7.20}$$

Assume that there is a nonsingular matrix

$$P = \begin{bmatrix} t_{1,1} & t_{1,2} & t_{1,3} & t_{1,4} & t_{1,5} \\ t_{2,1} & t_{2,2} & t_{2,3} & t_{2,4} & t_{2,5} \\ t_{3,1} & t_{3,2} & t_{3,3} & t_{3,4} & t_{3,5} \\ t_{4,1} & t_{4,2} & t_{4,3} & t_{4,4} & t_{4,5} \\ t_{5,1} & t_{5,2} & t_{5,3} & t_{5,4} & t_{5,5} \end{bmatrix}, \tag{7.21}$$

such that $A = P^{-1}\bar{A}P, B = P^{-1}\bar{B}, C = \bar{C}P$.

Then one have the following function

$$\begin{cases} PA - \bar{A}P = \mathbf{0}, \\ PB - P = \mathbf{0}, \\ C - \bar{C}T = \mathbf{0}. \end{cases} \tag{7.22}$$

However, it can be verified that there is no nonzero solution for the function in (7.21). This means that the Roesser model $(A, B, C, D; \boldsymbol{r})$ in (7.18) cannot be obtained from the Roesser model $(\bar{A}, \bar{B}, \bar{C}, D; \bar{\boldsymbol{r}})$ in (7.17) by a nonsingular transformation, i.e., structured similarity transformation or non-structured similarity transformation.

Therefore, it is needed to develop general transformations, which are defined as follows.

Definition 7.3. *The general transformations are operations that can transform the given n-D Roesser state-space model* $(\bar{A}, \bar{B}, \bar{C}, D; \bar{\boldsymbol{r}})$ *to a new n-D Roesser state-space model* $(A, B, C, D; \boldsymbol{r})$ *such that they have the same transfer matrix and the same total order but different partial orders.*

In Section 7.4, conditions and the corresponding procedure, by which an n-D Roesser state-space model $(\bar{A}, \bar{B}, \bar{C}, D; \bar{\boldsymbol{r}})$ can be transformed to a new n-D Roesser model by a general transformation, will be developed.

7.2.3 Exact Order Reduction Procedure

The following results are developed to show properties of the proposed two types of transformations, which will lead to an exact order reduction approach for n-D Roesser models.

Theorem 7.1. *If an n-D Roesser state-space model $(A, B, C, D; \boldsymbol{r})$ is obtained from a minimal noncommutative realization $(\bar{A}, \bar{B}, \bar{C}, D; \bar{\boldsymbol{r}})$ by a non-structured similarity transformation or a general transformation, then the n-D Roesser state-space models $(A, B, C, D; \boldsymbol{r})$ and $(\bar{A}, \bar{B}, \bar{C}, D; \bar{\boldsymbol{r}})$ have the same transfer matrix but different noncommutative transfer matrices.*

Proof. From the definition of non-structured similarity transformation and general transformation, one can see that the n-D Roesser models $(A, B, C, D; \boldsymbol{r})$ and $(\bar{A}, \bar{B}, \bar{C}, D; \bar{\boldsymbol{r}})$ have the same transfer matrix and $(A, B, C, D; \boldsymbol{r})$ is not obtained from $(\bar{A}, \bar{B}, \bar{C}, D; \bar{\boldsymbol{r}})$ by a structured similarity transformation. Moreover, we know that all the noncommutative minimal realizations are related by structure similarity transformations. Therefore, the n-D Roesser state-space models $(A, B, C, D; \boldsymbol{r})$ and $(\bar{A}, \bar{B}, \bar{C}, D; \bar{\boldsymbol{r}})$ have different noncommutative transfer matrices. $\square$

From Theorem 7.1, we can conclude that if one can transform a given minimal noncommutative realization $(\bar{A}, \bar{B}, \bar{C}, D; \bar{\boldsymbol{r}})$ to another n-D Roesser model $(A, B, C, D; \boldsymbol{r})$ by a non-structured similarity transformation or a general transformation, then the n-D Roesser model $(A, B, C, D; \boldsymbol{r})$ may be not minimal noncommutative realization, since the n-D Roesser state-space models $(A, B, C, D; \boldsymbol{r})$ and $(\bar{A}, \bar{B}, \bar{C}, D; \bar{\boldsymbol{r}})$ have different noncommutative transfer matrices. To intuitively show this fact, let us consider the following simple example.

Example 7.5. *Consider the 2-D Roesser model $(A, B, C, D; \boldsymbol{r})$ in (7.13), which is transformed from the minimal noncommutative 2-D Roesser model realization $(\bar{A}, \bar{B}, \bar{C}, D; \bar{\boldsymbol{r}})$ in (7.1) by a non-structured similarity matrix P in (7.12). One can compute that for* *the 2-D Roesser state-space model* $(A, B, C, D; \boldsymbol{r})$ *its noncommutative multidimensional reachability matrix is*

$$\tilde{\mathcal{C}}(A, B) = \left[\begin{array}{ccccccc} 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & 1 & 0 & 2 & 0 \end{array}\right] = \left[\begin{array}{c} \tilde{\mathcal{C}}_1 \\ \tilde{\mathcal{C}}_2 \end{array}\right]. \tag{7.23}$$

it can be verified that $\mathrm{rank}(R_1) = 1 < r_i = 2$. *Then, we know that the* 2*-D Roesser model* $(A, B, C, D; (2, 1))$ *in (7.13) is not a minimal noncommutative Roesser model.*

From the above discussion, one can conclude that if a minimal noncommutative n-D Roesser model $(A, B, C, D, ; \boldsymbol{r})$ can be transformed to another n-D Roesser model $(\bar{A}, \bar{B}, \bar{C}, D; \bar{\boldsymbol{r}})$ by a non-structured similarity transformation or a general transformation, the n-D Roesser model $(\bar{A}, \bar{B}, \bar{C}, D; \bar{\boldsymbol{r}})$ does not keep the property of the minimal noncommutative realization of the original Roesser model, even though the n-D Roesser model $(\bar{A}, \bar{B}, \bar{C}, D; \bar{\boldsymbol{r}})$ has the same total order with that of Roesser model $(A, B, C, D; \boldsymbol{r})$,

Then, a basic procedure for reducing the order of a given n-D Roesser state-space model is given in Procedure 7.1.

Procedure 7.1: Further Exact Order Reduction of an n-D Roesser model

Input : A given n-D Roesser model $(\boldsymbol{A}, \boldsymbol{B}, C, D; r)$;
Output: A lower-order n-D F-M model $(\hat{\boldsymbol{A}}, \hat{\boldsymbol{B}}, \hat{C}, \hat{D}; \hat{r})$;

Step 1: Reduce a given n-D Roesser model $(A, B, C, D; \boldsymbol{r})$ to a minimal noncommutative n-D Roesser model $(\bar{A}, \bar{B}, \bar{C}, D, \bar{\boldsymbol{r}})$;

Step 2: Construct a nonsingular matrix T in the form of (6.34) and compute

$$R \triangleq \left[\begin{array}{cc} R_1 & R_2 \end{array}\right] \triangleq T^{-1} \tag{7.24}$$

with $R_1 \in \mathbf{R}^{r \times \tilde{r}}$ and $R_2 \in \mathbf{R}^{(r \times \hat{r})}$;

Step 3: If the n-D Roesser model $(A, B, C, D; \boldsymbol{r})$ is not minimal noncommutative n-D Roesser model, then go to Step 1. Otherwise, quit the exact order reduction procedure;

return *the reduced-order Roesser model* $(\hat{\boldsymbol{A}}, \hat{\boldsymbol{B}}, \hat{C}, \hat{D}; \hat{r})$.

In the whole exact order reduction procedure, there are only two key points that need to be addressed. One is that how to reduce an n-D Roesser model $(A, B, C, D; \boldsymbol{r})$ to a minimal noncommutative n-D Roesser model $(\bar{A}, \bar{B}, \bar{C}, D; \bar{\boldsymbol{r}})$, which has been addressed in Section 5.2.2. The other is that how to transform a minimal noncommutative n-D

Roesser model $(\bar{A}, \bar{B}, \bar{C}, D; \bar{\boldsymbol{r}})$ to another n-D Roesser model $(A, B, C, D; \boldsymbol{r})$ by a non-structured similarity transformation or a general transformation. In Section 7.3 and 7.4, conditions and the corresponding procedures will be developed for transforming an n-D Roesser state-space model $(\bar{A}, \bar{B}, \bar{C}, D; \bar{\boldsymbol{r}})$ to a new n-D Roesser model $(A, B, C, D; \boldsymbol{r})$ by a non-structured similarity transformation and a general transformation, respectively.

7.3 Non-Structured Similarity Transformation

In this section, a non-structured similarity transformation will be introduced, by which a new equivalent n-D Roesser model $(A, B, C, D; \boldsymbol{r})$ can be obtained from a given n-D Roesser model $(A, B, C, D; \boldsymbol{r})$ such that $(A, B, C, D; \boldsymbol{r})$ and have the same transfer matrix but different noncommutative transfer matrices.

Theorem 7.2. *For a given n-D Roesser state-space model $(A, B, C, D, \boldsymbol{r})$, suppose that there is a pair of indices (ξ, η) such that*

$$a_{\xi,j} = 0 \quad \textit{for all } j \neq \eta, \xi; \tag{7.25a}$$

$$\textit{and} \quad a_{k,\eta} = 0 \quad \textit{for all } k \neq \eta, \xi; \tag{7.25b}$$

$$\textit{and} \quad b_{\xi,j} = 0 \quad \textit{for all } j; \tag{7.25c}$$

$$\textit{and} \quad c_{k,\eta} = 0 \quad \textit{for all } k. \tag{7.25d}$$

Then, letting

$$A = T^{-1}AT, \quad B = T^{-1}B, \quad C = CT, \tag{7.26}$$

with

$$T = \begin{array}{c} \begin{array}{ccccc} & \downarrow \eta th & & \downarrow \xi th & \end{array} \\ \begin{bmatrix} I & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \gamma & \mathbf{0} & 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 1 & \mathbf{0} & 0 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & I \end{bmatrix} \end{array} \begin{array}{l} \\ \\ \leftarrow \eta th \\ \\ \leftarrow \xi th \\ \\ \end{array}, \tag{7.27}$$

and $\gamma = \frac{a_{\eta,\eta} - a_{\xi,\xi}}{a_{\xi,\eta}}$, *we have*

$$CZ(I_r - AZ)^{-1}B + D = CZ(I_r - AZ)^{-1}B + D. \tag{7.28}$$

Remark 7.1. *The key point here is that the ξth and ηth columns of A belong to different blocks A_{-i} in (4.5). Otherwise, T in (7.27) is just a normal structured similar transformation matrix.*

By Theorem 7.2, it is known that for a given n-D Roesser model $(A, B, C, D, ; \boldsymbol{r})$, if the ξth row of A and B satisfies the conditions (7.25a) and (7.25c), respectively; and the ηth column of A and C satisfies the conditions (7.25b) and (7.25d), then one can obtain a new n-D Roesser model $(A, B, C, D; \boldsymbol{r})$ by using a non-structured similar transformation matrix T in (7.27) to the given n-D Roesser model. Since the conditions of Theorem 7.2 utilize the row block A_{i_1-} and column block A_{-i_2}, we can then call this non-structured similarity transformation obtained by considering the relationship between column block and row block.

The following example is given to show the effectiveness of Theorem 7.2.

Example 7.6. *Consider the 2-D Roesser state-space model of $(A, B, C, D; \boldsymbol{r})$ of (7.1) in Example 7.1 which is minimal in the non-commutative case, but $(\xi, \eta) = (2, 3)$ is a pair according to Theorem 7.2.*

7.4 General Transformation

In the previous section, the non-structured similarity transformations have been proposed based on the relationship between a row block and a column block related to different variables. In this section, the general transformations will be proposed based on the relation between column (or row) blocks by introducing the well defined n-D system matrix corresponding n-D Roesser state-space models.

7.4.1 Notion of the n-D System matrix

Let $\mathcal{M}(p, q)$ be the class of $(r + p) \times (r + q)$ matrices where p and q are fixed integers and r ranges over all positive integers. Then, if $M \in \mathcal{M}(p, q)$, it can be partitioned as

$$M = \left[\begin{array}{c|c} M_{1,1} & M_{1,2} \\ \hline M_{2,1} & M_{2,2} \end{array}\right] \tag{7.29}$$

with $M_{1,1}$ being invertible and $M_{2,2} \in \mathbf{R}^{p\times q}$ and then a map on M can be defined as

$$\mathcal{R}(M) = -M_{2,1}M_{1,1}^{-1}M_{1,2} + M_{2,2}. \tag{7.30}$$

Then, we have the following result.

Lemma 7.1. *Let L and R be matrices of suitable sizes obtained as products of some elementary matrices and/or elementary rational polynomial matrices, and satisfy*

$$L = \begin{bmatrix} K & \mathbf{0} \\ \bar{K} & I_p \end{bmatrix}, \quad R = \begin{bmatrix} N & \bar{N} \\ \mathbf{0} & I_q \end{bmatrix}, \tag{7.31}$$

with nonsingular matrix K and N. Then we have

$$\mathcal{R}(M) = \mathcal{R}(LMR). \tag{7.32}$$

Proof. With attention that

$$\begin{aligned} &LMR \\ =&\left[\begin{array}{c|c} KM_{1,1}N & KM_{1,1}\bar{N}+KM_{1,2} \\ \hline \bar{K}M_{1,1}N+M_{2,1}N & \bar{K}M_{1,1}\bar{N}+M_{2,1}\bar{N}+\bar{K}M_{1,2}+M_{2,2} \end{array}\right], \end{aligned} \tag{7.33}$$

then, we have

$$\begin{aligned} \mathcal{R}(LMR) =& -(\bar{K}M_{1,1}N + M_{2,1}N)(KM_{1,1}N)^{-1}(KM_{1,1}\bar{N} + KM_{1,2}) \\ &+ (\bar{K}M_{1,1}\bar{N} + M_{2,1}\bar{N} + \bar{K}M_{1,2} + M_{2,2}) \\ =& - M_{2,1}M_{1,1}^{-1}M_{1,2} + M_{2,2} = \mathcal{R}(M). \end{aligned} \tag{7.34}$$

□

Definition 7.4. *The n-D system matrix M corresponding to an n-D Roesser state-space model $(A, B, C, D; \boldsymbol{r})$ can be defined as*

$$M = \begin{bmatrix} I - AZ & B \\ -CZ & D \end{bmatrix} \tag{7.35}$$

where $Z = diag\{z_1 I_{r_1}, \ldots, z_n I_{r_n}\}$.

Observing the system matrix M defined in Definition 7.4, the structural properties of the n-D system matrix for an n-D Roesser model in the form of (7.35) can be found as follows.

(a) Each of the diagonal entries except those in D must be a linear 1-D polynomial in certain z_i, $i \in \{1, \dots, n\}$, with the constant term being 1;

(b) Each of the nondiagonal entries in the first r columns must be zero or a linear monomial in certain z_i, $i \in \{1, \dots, n\}$;

(c) Each of the entries in the last q columns must be a constant;

(d) The entries in the same column contain only the same variable z_i, $i \in \{1, \dots, n\}$;

(e) All the first r columns are sorted in the order of $z_1, \dots, z_n$.

It should be noted that an n-D system matrix M in the form of (7.35) and an n-D Roesser state-space model $(A, B, C, D; \boldsymbol{r})$ have a one-to-one relationship. That is, for an n-D Roesser model $(A, B, C, D; \boldsymbol{r})$, only one n-D system matrix M in the form of (7.35) can be obtained; for an n-D system matrix M in the form of (7.35), one can also extract only one n-D Roesser state-space model. Moreover, the partial order r_i of an n-D Roesser state-space model, $i \in \{1, \dots, n\}$, is equal to the number of column vectors with variables z_i in M. Therefore, we can call r_i and r the partial order and total order of the n-D system matrix M, respectively, where r_i is equal to the number of column vectors with variables z_i and $r = r_1 + \dots, r_n$.

Definition 7.5. *An n-D pseudo-system matrix $\check{M}$ is a polynomial matrix in the form of*

$$\check{M} = \begin{bmatrix} F - \check{A}\check{Z} & \check{B} \\ -\check{C}\check{Z} & D \end{bmatrix} \tag{7.36}$$

where F is a nonsingular matrix, $\check{Z} = diag\{z_{i_1} I_{\check{r}_1}, \dots, z_{i_m} I_{\check{r}_m}\}$, $i_t \in \{1, \dots, n\}$, $\check{r} = \check{r}_1 + \dots + \check{r}_m$.

The structure properties of an n-D pseudo-system matrix $\check{M}$ in the form of (7.35) can be found as follows:

(a) Each of the entries in the first r columns must be a linear 1-D polynomial in certain z_i, $i \in \{1, \dots, n\}$.

(b) Each of the entries in the last q columns must be a constant;

(c) The entries in the same column contain only the same variable z_i, $i \in \{1, \dots, n\}$.

As with the same definition in the n-D system matrix M in (7.35), we can call $\check{r}_{oi}$ and $\check{r_o}$ the partial order and total order of an n-D pseudo-system matrix $\check{M}$ in (7.36), respectively, where $\check{r}_{oi}$ is equal to the number of column vectors with variables z_i and $r_{oi} = \check{r}_{o1} + \ldots, \check{r}_{on} = \check{r}$.

Lemma 7.2. *There is a pair of appropriate matrices L and R in the form of (7.31) which can transform the n-D pseudo-system matrix $\check{M}$ in the form of (7.36) to an n-D system matrix M in the form of*

$$M = \begin{bmatrix} I - AZ & B \\ -CZ & D \end{bmatrix} \tag{7.37}$$

where $Z = diag\{z_1 I_{r_1}, \ldots, z_n I_{r_n}\}$, $r_i = \check{r}_{oi}$, $i = 1, \ldots, n$.

Proof. Let N be a permutation matrix such that the columns in $\check{Z}N$ are sorted in order of $z_1, \ldots, z_n$, which gives that $N^{\mathrm{T}}\check{Z}N = N\check{Z}N = Z$ with $Z = \mathrm{diag}\{z_1 I_{r_1}, \ldots, z_n I_{r_n}\}$. Note that $r_i = \check{r}_{oi}$, $i = 1, \ldots, n$, where $\check{r}_{oi}$ is the number of column vectors in $\check{Z}$ with variable z_i.

We then construct

$$\check{L} = \begin{bmatrix} NF^{-1} & \mathbf{0} \\ \mathbf{0} & I_p \end{bmatrix}, \quad \check{R} = \begin{bmatrix} N & \mathbf{0} \\ \mathbf{0} & I_q \end{bmatrix}. \tag{7.38}$$

With attention that a permutation matrix N has the properties $N = N^{-1} = N^{\mathrm{T}}$, we have

$$\begin{aligned} M \triangleq \check{L}\check{M}\check{R} &= \begin{bmatrix} I - NF^{-1}\check{A}\check{Z}N & NF^{-1}\check{B} \\ -\check{C}\check{Z}N & D \end{bmatrix} \\ &= \begin{bmatrix} I - NF^{-1}\check{A}NZ & NF^{-1}\check{B} \\ -\check{C}NZ & D \end{bmatrix}. \end{aligned}$$

which is an n-D system matrix with $A = NF^{-1}\check{A}N$, $B = NF^{-1}\check{B}$, $C = \check{C}N$.

This means that there is a pair of matrices $\check{L}$ and $\check{R}$ which can always transform a pseudo-system matrix $\tilde{M}$ in the form of (7.36) to an n-D system matrix M in the form of (7.37). □

7.4.2 Conditions and the Corresponding Procedure for General Transformation

Theorem 7.3. *For a given minimal noncommutative n-D Roesser model $(A, B, C, D; \boldsymbol{r})$ and its n-D system matrix M in the form of (7.35), if there is a pseudo-system matrix $\check{M}$*

obtained by

$$\check{M} = LMR \tag{7.39}$$

with L and R in the form of (7.31) such that $\check{M}$ and M have the same total order but different partial orders, then, there is an equivalent n-D Roesser model $(A, B, C, D; \boldsymbol{r})$ corresponding to $(A, B, C, D; \boldsymbol{r})$. Here, Roesser models $(A, B, C, D; \boldsymbol{r})$ and $(A, B, C, D; \boldsymbol{r})$ have the same transfer matrix but different noncommutative transfer matrices.

Proof. By Lemma 7.2, one can see that the pseudo-system matrix $\check{M}$ in (7.36) can be transformed to an n-D system matrix M in the form of (7.37) and have a relationship that

$$\mathcal{R}(\check{M}) = \mathcal{R}(M) = CZ(I - AZ)^{-1}B + D \tag{7.40}$$

with $Z = \mathrm{diag}\{z_1 I_{r_1}, \ldots, z_n I_{r_n}\}$, $r_i = \check{r}_{oi}$, $i = 1, \ldots, n$.

Note that $\check{M}$ is obtained from the n-D system matrix M in (7.31) corresponding to the given n-D Roesser model $(A, B, C, D; \boldsymbol{r})$ and they have a relationship that

$$CZ(I - AZ)^{-1}B + D = \mathcal{R}(M) = \mathcal{R}(\check{M}) \tag{7.41}$$

with $\tilde{r}_{oi} \neq r_i$, $i \in \{1, \ldots, n\}$.

Using equations (7.40) and (7.41), we have

$$CZ(I - AZ)^{-1}B + D = CZ(I - AZ)^{-1}B + D, \tag{7.42}$$

and then the Roesser state-space model $(A, B, C, D; \boldsymbol{r})$ is equivalent to the given Roesser state-space model $(A, B, C, D; \boldsymbol{r})$.

From $r_i \neq r_i$, $i \in \{1, \ldots, n\}$, it can be inferred that the Roesser models $(A, B, C, D; \boldsymbol{r})$ and $(A, B, C, D; \boldsymbol{r})$ have different noncommutative transfer matrices. □

Assume that λ_i is one eigenvalues of matrix $A_{i,i}$, $i \in \{1, \ldots, n\}$. Define

$$\mathcal{N}_c \triangleq \left[\begin{array}{c|c|c} A_{-i_1} - \lambda_{i_1} E_{-i_1} & -E_{-i_2} & \mathbf{0} \\ C_{i_1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & A_{-i_2} & A_{-i_1} - \lambda_{i_1} E_{-i_1} \\ \mathbf{0} & C_{i_2} & C_{i_1} \end{array}\right], \tag{7.43}$$

and

$$\mathcal{N}_r \triangleq \begin{bmatrix} A_{i_1-} - \lambda_{i_1} E_{i_1-} & B_{i_1} & \mathbf{0} & \mathbf{0} \\ I_{i_2-} & \mathbf{0} & A_{i_2-} & B_{i_2} \\ \mathbf{0} & \mathbf{0} & A_{i_1-} - \lambda_{i_1} E_{i_1-} & B_{i_1} \end{bmatrix}, \tag{7.44}$$

where A_{-i_1}, A_{-i_2}, C_{i_1}, C_{i_2}, A_{i_1-}, A_{i_2-}, B_{i_1} and B_{i_2} are defined in (4.5).

Lemma 7.3. *For the matrix $\mathcal{N}_c$ defined in (7.43) of a given minimal noncommutative n-D Roesser model $(A, B, C, D; \boldsymbol{r})$, if there is a nonzero vector*

$$\begin{aligned} \boldsymbol{\mu} &\triangleq \begin{bmatrix} \boldsymbol{\mu}_1 & \boldsymbol{\mu}_2 & \boldsymbol{\mu}_3 \end{bmatrix}^{\mathrm{T}} \\ &= \left[\begin{array}{ccc|ccc|ccc} \mu_{11} & \dots & \mu_{1,r_{i_1}} & \mu_{21} & \dots & \mu_{2,r_{i_2}} & \mu_{31} & \dots & \mu_{3,r_{i_1}} \end{array} \right]^{\mathrm{T}}, \end{aligned} \tag{7.45}$$

such that such that $\mathcal{N}_c \boldsymbol{\mu} = \mathbf{0}$, i.e., the matrix $\mathcal{N}_c$ is not full column rank, then the vector $\boldsymbol{\mu}$ is not a zero vector.

Proof. Equation $\mathcal{N}_c \boldsymbol{\mu} = \mathbf{0}$ can be written in terms of their partitioned matrices as

$$(A_{-i_1} - \lambda_{i_1} E_{-i_1}) \boldsymbol{\mu}_1 - E_{-i_2} \boldsymbol{\mu}_2 = \mathbf{0}, \tag{7.46a}$$

$$C_{i_1} \boldsymbol{\mu}_1 = \mathbf{0}, \tag{7.46b}$$

$$A_{-i_2} \boldsymbol{\mu}_2 + (A_{-i_1} - \lambda_{i_1} E_{-i_1}) \boldsymbol{\mu}_3 = \mathbf{0}, \tag{7.46c}$$

$$C_{i_2} \boldsymbol{\mu}_2 + C_{i_1} \boldsymbol{\mu}_3 = \mathbf{0}., \tag{7.46d}$$

We assume that $\boldsymbol{\mu}_1 = \mathbf{0}$, then it can be obtained by equation (7.46a) that $\boldsymbol{\mu}_2 = \mathbf{0}$. Substituting $\boldsymbol{\mu}_2 = \mathbf{0}$ into equations (7.46c) and (7.46d), one can get

$$(A_{-i_1} - \lambda_{i_1} E_{-i_1}) \boldsymbol{\mu}_3 = \mathbf{0}, \tag{7.47a}$$

$$C_1 \boldsymbol{\mu}_3 = \mathbf{0}. \tag{7.47b}$$

This contradicts the Theorem 4.1. □

Theorem 7.4. *For the matrix $\mathcal{N}_c$ defined in (7.43) of a given minimal commutative n-D Roesser model $(A, B, C, D; \boldsymbol{r})$, if there is a nonzero vector*

$$\begin{aligned} \boldsymbol{\mu} &\triangleq \begin{bmatrix} \boldsymbol{\mu}_1 & \boldsymbol{\mu}_2 & \boldsymbol{\mu}_3 \end{bmatrix}^{\mathrm{T}} \\ &= \left[\begin{array}{ccc|ccc|ccc} \mu_{11} & \dots & \mu_{1,r_{i_1}} & \mu_{21} & \dots & \mu_{2,r_{i_2}} & \mu_{31} & \dots & \mu_{3,r_{i_1}} \end{array} \right]^{\mathrm{T}} \end{aligned} \tag{7.48}$$

such that $\mathcal{N}\boldsymbol{\mu} = \mathbf{0}$, i.e., the matrix $\mathcal{N}_c$ is not full column rank. Then there is a pseudo-system matrix $\check{M}$ obtained by

$$\check{M} = LMR \tag{7.49}$$

with

$$L \triangleq I_{r+p}, \ \ R \triangleq \begin{bmatrix} N & \mathbf{0} \\ \mathbf{0} & I_q \end{bmatrix}. \tag{7.50}$$

where N is an identity matrix with its $\bar{t}$th column replaced by

$$\boldsymbol{\xi} \triangleq \frac{1}{(1-\lambda_{i_1} z_1)} \left(E_{-i_1}(\boldsymbol{\mu}_1 + z_{i_2}\boldsymbol{\mu}_3) + E_{-i_2} z_1 \boldsymbol{\mu}_2\right) \tag{7.51}$$

where $\bar{t} = \sum_{k=0}^{i_1-1}(r_i) + t$ with $r_0 = 0$ and $\mu_{1,t} \neq 0$, such that $\check{M}$ and M have the same total order but different partial order.

Proof. Without of generality, we assume that $i_1 = 1$, $i_2 = 2$, $\mu_{1,1} \neq 0$ then set $t = 1$, since for the other case, the proof is similar and thus omitted.

Equation (7.46c) can be written in terms of their partitioned matrices as

$$A_{1,2}\boldsymbol{u}_2 + (A_{1,1} - \lambda_1 I_{r_1})\boldsymbol{u}_3 = 0, \tag{7.52a}$$

$$A_{k,2}\boldsymbol{u}_2 + A_{k,1}\boldsymbol{u}_3 = 0 \quad \text{if} \quad k \neq 1. \tag{7.52b}$$

and equation (7.46a) gives that

$$(A_{1,1} - \lambda_1 I_{r_1})\boldsymbol{u}_1 = \mathbf{0}, \tag{7.53a}$$

$$A_{2,1}\boldsymbol{u}_1 - I_{r_2}\boldsymbol{u}_2 = \mathbf{0}, \tag{7.53b}$$

$$A_{k,1}\boldsymbol{u}_1 = \mathbf{0} \quad \text{if} \quad k \neq 1, \quad k \neq 2. \tag{7.53c}$$

Therefore, from equation (7.52a) and (7.53a), it can be obtained that

$$\begin{aligned}
&(I_{r_1} - A_{1,1}z_1)(\boldsymbol{\mu}_1 + z_2\boldsymbol{\mu}_3) + (-A_{1,2}z_2)z_1\boldsymbol{\mu}_2 \\
&= ((I_{r_1} - A_{1,1}z_1)\boldsymbol{\mu}_1) + ((I_{r_1} - A_{1,1}z_1)z_2\boldsymbol{\mu}_3 - A_{1,2}z_2 z_1\boldsymbol{\mu}_2) \\
&= (I_{r_1} - \lambda_1 z_1 I_{r_1})\boldsymbol{\mu}_1 + (I_{r_1} - \lambda_1 z_1 I_{r_1})z_2\boldsymbol{\mu}_1 \\
&= (1 - \lambda_1 z_1)(\boldsymbol{\mu}_1 + z_2\boldsymbol{\mu}_3).
\end{aligned} \tag{7.54}$$

By equation (7.52b) and (7.53b), it can be got that

$$-A_{2,1}z_1(\boldsymbol{\mu}_1 + z_2\boldsymbol{\mu}_3) + (I_{r_2} - A_{2,2}z_2)z_1\boldsymbol{\mu}_2 = \mathbf{0}. \tag{7.55}$$

Equations (7.52b) and (7.53c) give that

$$-A_{k,1}z_1(\boldsymbol{\mu}_1 + z_2\boldsymbol{\mu}_3) + (-A_{k,2}z_2)z_1\boldsymbol{\mu}_2 = \mathbf{0} \tag{7.56}$$

with $k \neq 1, 2$, and equations (7.46b) and (7.46d) give that

$$C_1z_1(\boldsymbol{\mu}_1 + z_2\boldsymbol{\mu}_3) + (C_2z_2)z_1\boldsymbol{\mu}_2 = \mathbf{0}. \tag{7.57}$$

Then, from equations (7.54), (7.55), (7.56) and (7.57) it can be obtained that multiplication of the system matrix M of related n-D Roesser model from right by $\boldsymbol{\xi}$ gives

$$\begin{aligned} M\boldsymbol{\xi} &= \begin{bmatrix} I_{r_1}-A_{1,1}z_1 & -A_{1,2}z_2 & \ldots & -A_{1,n}z_n & B_1 \\ -A_{2,1}z_1 & I_{r_2}-A_{2,2}z_2 & \ldots & -A_{2,n}z_n & B_2 \\ \vdots & \vdots & & \vdots & \vdots \\ -A_{n,1}z_1 & -A_{n,2}z_2 & \ldots & I_{r_n}-A_{n,n}z_n & B_n \\ C_1z_1 & C_2z_2 & \ldots & C_nz_n & D \end{bmatrix} \begin{bmatrix} \frac{\boldsymbol{\mu}_1+z_2\boldsymbol{\mu}_3}{(1-\lambda_1z_1)} \\ \frac{z_1\boldsymbol{\mu}_2}{(1-\lambda_1z_1)} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \\ &= \begin{bmatrix} (\boldsymbol{\mu}_1 + z_2\boldsymbol{\mu}_3)^{\mathrm{T}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}^T \\ &= E_{-1}(\boldsymbol{\mu}_3 + z_2\boldsymbol{\mu}_1) \end{aligned} \tag{7.58}$$

Noting that L is an identity matrix and R is an identity matrix with its first column replaced by $\boldsymbol{\xi}$, then it can be seen that the multiplication LMR will keep the columns of M unchanged except the first column in which the variable z_1 is changed to z_2, $z_1 \neq z_2$. Thus, the n-D pseudo-system matrix $\check{M}$ in the form of (7.36) has been obtained from the n-D system matrix M in the form of (7.31) such that $\check{r}_1 = r_1 - 1$ and $\check{r}_2 = r_2 + 1$, i.e., $\check{M}$ and M have different partial orders. $\square$

Theorem 7.4 gives a sufficient condition for obtaining a pseudo-system matrix $\check{M}$ from an n-D system matrix M corresponding to a given n-D Roesser model $(A, B, C, D; \boldsymbol{r})$, that is if the matrix $\mathcal{N}_c$ defined in (7.43) is not full rank, such that M and $\check{M}$ have the same order but different partial order. Moreover, Theorem 7.3 shows that for a given n-D Roesser state-space model $(A, B, C, D; \boldsymbol{r})$, if there is a pseudo-system matrix $\check{M}$ obtained from M such that $\check{M}$ and M have the same total order but different partial orders, then there is an equivalent n-D Roesser model $(A, B, C, D; \boldsymbol{r})$ corresponding to $(A, B, C, D; \boldsymbol{r})$ such that $(A, B, C, D; \boldsymbol{r})$ and $(A, B, C, D; \boldsymbol{r})$ have the same transfer matrix but different noncommutative matrices. In what follows, a procedure is given to obtain an equivalent n-D Roesser model by the general transformation.

Procedure 7.2: General Transformation

Input : A given n-D Roesser model $(\boldsymbol{A}, \boldsymbol{B}, C, D; r)$;
Output: An equivalent n-D Roesser model $(\hat{\boldsymbol{A}}, \hat{\boldsymbol{B}}, \hat{C}, \hat{D}; \hat{r})$;

Step 1: If the matrix N_c in (7.43) is not full rank, transform the n-D system matrix M corresponding to the given n-D Roesser model $(A, B, C, D; \boldsymbol{r})$ to a pseudo-system matrix $\check{M}$ construct L and R in the from of (7.50). Set $(A, B, C, D; \boldsymbol{r}) = (A, B, C, D; \boldsymbol{r})$.
return $(\hat{\boldsymbol{A}}, \hat{\boldsymbol{B}}, \hat{C}, \hat{D}; \hat{r})$.

Example 7.7. *Consider the Roesser model* $(A, B, C, D) = (\bar{A}, \bar{B}, \bar{C}, \bar{D})$ *with order* $(r_1, r_2) = (3, 2)$ *in (7.17), whose first standard system matrix* $M(z)$

$$M(z) = \begin{bmatrix} I - AZ & B \\ -CZ & D \end{bmatrix}. \tag{7.59}$$

Note that this 2-D Roesser model can not be reduced by the proposed order reduction the common invariant subspace appraoch and existing order reduction techniques in [4, 29–32]. However, we will show that, this model is equivalent to another Roesser model which can be further reduced by the proposed common invariant subspace reduction.

For this model, we have

$$I_{-1} = \begin{bmatrix} I_3 \\ \mathbf{0}_{2\times 3} \end{bmatrix} = \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right], \quad I_{-2} = \begin{bmatrix} \mathbf{0}_{2\times 2} \\ I_2 \end{bmatrix} = \left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \hline 1 & 0 \\ 0 & 1 \end{array}\right], \tag{7.60}$$

and

$$A_{-1} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \\ 1 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_{-2} = \begin{bmatrix} 0 & -2 \\ 1 & 2 \\ 0 & 1 \\ -1 & 0 \\ 0 & -\frac{3}{2} \end{bmatrix}. \tag{7.61}$$

Step 1: Note that $\lambda_1 = 2$ is one eigenvalue of A_{11}. Let $i_1 = 1$ and $i_2 = 2$, using (7.43) we have

$$\mathcal{N} \triangleq \left[\begin{array}{c|c|c} 2I_{-1} - A_{-1} & -A_{-2} & 0 \\ C_1 & C_2 & 0 \\ 0 & I_{-2} & 2I_{-1} - A_{-1} \\ 0 & 0 & C_1 \end{array}\right]$$

$$= \left[\begin{array}{ccc|cc|ccc} 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 1 & -2 & -1 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ -1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{3}{2} & 0 & 0 & 0 \\ \hline -1 & 2 & -2 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -2 & -1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & -1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -2 \end{array}\right], \tag{7.62}$$

which has not full rank. That is, there is a vector

$$\boldsymbol{\mu} = \left[\begin{array}{ccc} \boldsymbol{\mu}_1 & \boldsymbol{\mu}_2 & \boldsymbol{\mu}_3 \end{array}\right] = \left[\begin{array}{ccc|cc|ccc} 1 & 1 & 1 & -1 & 0 & 2 & 2 & 1 \end{array}\right]^{\mathrm{T}} \tag{7.63}$$

such that $\mathcal{N}\boldsymbol{\mu} = \mathbf{0}$. Note that $\mu_{31} = 2 \neq 0$, so set $t = 1$. Then, we have

$$\boldsymbol{\xi} = \frac{1}{1 - 2z_1}\left(I_{1-}(\boldsymbol{\mu}_3 + z_2\boldsymbol{\mu}_1) + I_{2-}\boldsymbol{\mu}_2 z_1\right) = \begin{bmatrix} \frac{2+z2}{1-2z_1} \\ \frac{2+z2}{1-2z_1} \\ \frac{1+z2}{1-2z_1} \\ \frac{-z_1}{1-2z_1} \\ 0 \end{bmatrix}$$

and construct N as

$$N = \begin{bmatrix} \frac{z_2+2}{1-2z_1} & 0 & 0 & 0 & 0 \\ \frac{z_2+2}{1-2z_1} & 1 & 0 & 0 & 0 \\ \frac{z_2+1}{1-2z_1} & 0 & 1 & 0 & 0 \\ \frac{-z_1}{1-2z_1} & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{7.64}$$

which is an identity matrix with its first column replaced by $\boldsymbol{\xi}$.

Then, as in (7.50) we have

$$L \triangleq I_7,\ R \triangleq \begin{bmatrix} N & \\ & I_2 \end{bmatrix}. \tag{7.65}$$

and

$$\begin{aligned} \tilde{M}(z) =& LM(z)R \\ =& \left[\begin{array}{ccc|cc|cc} z_2+2 & 0 & 0 & 0 & 2z_2 & 2 & 2 \\ z_2+2 & 1-z_1 & -2z_1 & -z_2 & -2z_2 & -6 & -4 \\ z_2+1 & 0 & 1-2z_1 & 0 & -z_2 & 0 & 0 \\ 0 & z_1 & z_1 & z_2+1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & \frac{3z_2}{2}+1 & 1 & 1 \\ 0 & 2z_1 & -2z_1 & -z_2 & 0 & 0 & 0 \\ 0 & 0 & -2z_1 & -z_2 & 0 & 0 & 0 \end{array}\right]. \end{aligned} \tag{7.66}$$

It should be noted that the order of the system matrix $\tilde{M}(z)$ in (7.66) is $(\tilde{r}_{o1}, \tilde{r}_{o2}) = (2,3)$, and the order of the system matrix $M(z) = \bar{M}(z)$ in (7.59) corresponding the Roesser model in (7.4) is $(r_1, r_2) = (3,2)$. That is, the system matrix $\tilde{M}(z)$ in (7.66) and $M(z) = \bar{M}(z)$ in (7.59) have different partial orders.

Step 2: Comparing the obtained system matrix $\tilde{M}$ in (7.66) to the system matrix $\tilde{M}$ in (7.36), we have

$$F = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \tag{7.67a}$$

$$\tilde{Z} = \begin{bmatrix} z_2 & 0 & 0 & 0 & 0 \\ 0 & z_1 & 0 & 0 & 0 \\ 0 & 0 & z_1 & 0 & 0 \\ 0 & 0 & 0 & z_2 & 0 \\ 0 & 0 & 0 & 0 & z_2 \end{bmatrix}. \tag{7.67b}$$

The permutation matrix

$$\tilde{N} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{7.68}$$

can make the columns in $\tilde{Z}\tilde{N}$ in a sort order. Then, using (7.38) we have

$$\tilde{L} = \begin{bmatrix} \tilde{N}F^{-1} & \mathbf{0} \\ \mathbf{0} & I_2 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\tilde{R} = \begin{bmatrix} \tilde{N} & \mathbf{0} \\ \mathbf{0} & I_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \tag{7.69}$$

and can obtain a standard system matrix

$$\begin{aligned} \bar{M}(z) =& \tilde{L}\tilde{M}(z)\tilde{R} \\ =& \begin{bmatrix} 1-z_1 & -2z_1 & 0 & -z_2 & -4z_2 & -8 & -6 \\ 0 & 1-2z_1 & \frac{z_2}{2} & 0 & -2z_2 & -1 & -1 \\ 0 & 0 & \frac{z_2}{2}+1 & 0 & z_2 & 1 & 1 \\ z_1 & z_1 & 0 & z_2+1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & \frac{3z_2}{2}+1 & 1 & 1 \\ 2z_1 & -2z_1 & 0 & -z_2 & 0 & 0 & 0 \\ 0 & -2z_1 & 0 & -z_2 & 0 & 0 & 0 \end{bmatrix}. \end{aligned} \tag{7.70}$$

Step III: From $\bar{M}(z)$ of (7.70), one can obtain an Roesser model $(\bar{A}, \bar{B}, \bar{C}, D)$ of (7.4) as

$$\bar{A} = \left[\begin{array}{cc|ccc} 1 & 2 & 0 & 1 & 4 \\ 0 & 2 & -\frac{1}{2} & 0 & 2 \\ \hline 0 & 0 & -\frac{1}{2} & 0 & -1 \\ -1 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -\frac{3}{2} \end{array}\right], \quad \bar{B} = \left[\begin{array}{cc} -8 & -6 \\ -1 & -1 \\ \hline 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{array}\right],$$
$$\bar{C} = \left[\begin{array}{cc|ccc} 2 & -2 & 0 & -1 & 0 \\ 0 & -2 & 0 & -1 & 0 \end{array}\right], \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \tag{7.71}$$

It is seen that the order of the obtained Roesser model (7.71) is $(\bar{r}_1, \bar{r}_2) = (2, 3)$, which is different from the order $(r_1, r_2) = (3, 2)$ of the given Roesser model in (7.4).

7.5 Examples

In this section, examples are given to illustrate the effectiveness of the proposed approach.

Example 7.8. *Consider the following LFR given by Example 3.2 on page 33 in [89]:*

$$M_3 = \left[\begin{array}{cccc|c} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 1 & 0 \end{array}\right], \tag{7.72}$$

$$\Delta_3 = \begin{bmatrix} \delta_1 & 0 & 0 & 0 \\ 0 & \delta_1 & 0 & 0 \\ 0 & 0 & \delta_2 & 0 \\ 0 & 0 & 0 & \delta_2 \end{bmatrix}, \tag{7.73}$$

which cannot be reduced by the exact reduction techniques given in [4]. However, an order reduction of the LFR (M_3, Δ_3) can be achieved by the new proposed reduction approaches as follows:

$$\begin{aligned} M_3 &= \left[\ 0\ \right], \\ \Delta_4 &= \varnothing, \end{aligned} \tag{7.74}$$

with order $r = 0$.

Example 7.9. *Consider the following 4-D transfer function matrix:*

$$H(z_1, z_2, z_3, z_4) = \begin{bmatrix} \dfrac{z_1 z_2 z_3}{1 + z_1 + 2z_2} - \dfrac{2z_1 z_4 z_3}{1 - 2z_3 - z_4} \\ \\ \dfrac{2z_2 z_1 z_3}{1 + z_1 + 2z_2} + \dfrac{z_4 z_1 z_3}{1 - 2z_3 - z_4} \end{bmatrix}. \tag{7.75}$$

Applying LFR toolbox [4] yields the following Roesser model with order $r = 20$:

$$A = \left[\begin{array}{cccccc|cccc|cccccc|cccc}
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & -1 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 1 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 1
\end{array}\right], \quad B = \left[\begin{array}{c}
0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ \hline 0 \\ 1 \\ 0 \\ 1 \\ \hline 1 \\ -1 \\ 1 \\ 1 \\ 1 \\ -1 \\ \hline 0 \\ 1 \\ 0 \\ -1
\end{array}\right], \tag{7.76}$$

$$C = \left[\begin{array}{cccccc|cccc|cccccc|cccc}
1 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{array}\right], \quad D = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

By the proposed reduction method, we obtain the following much-lower Roesser model:

$$A = \left[\begin{array}{ccc|c|cc|cc}
0 & 0 & 0 & 0 & 1 & 0 & 2 & 0 \\
0 & -1 & 0 & -2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & -\frac{1}{2} & 0 \\
\hline
0 & -1 & 0 & -2 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2 & 0 & -1 \\
\hline
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -2 & 0 & 1
\end{array}\right], \quad B = \left[\begin{array}{c}
0 \\ 1 \\ 0 \\ \hline 1 \\ \hline 0 \\ -1 \\ \hline 0 \\ 1
\end{array}\right] \tag{7.77}$$

$$C = \left[\begin{array}{ccc|c|cc|cc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & 0 & 0 & 0 & 0 & 0
\end{array}\right], \quad D = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The comparisons with the existing approaches are shown in Table 7.1.

7.6 Proofs of the Main Results

Before give the main proofs, we need the following result.

Table 7.1: Comparison of the reduced orders for the Roesser model in 7.76.

Methods	**partial Orders**	**Order**
Method of [23]	(4,3,4,4)	15
LFR Toolbox [4]	(4,3,2,3)	12
Our method	(3,1,2,2)	8

Lemma 7.4. *The matrices $M, \tilde{M}, \check{M}, \acute{M}, \in \mathcal{M}(p,q)$ in the form of*

$$M = \left[\begin{array}{ccc|c} M_{1,1,1} & M_{1,1,2} & M_{1,1,3} & M_{1,2,1} \\ \mathbf{0} & M_{1,1,5} & \mathbf{0} & \mathbf{0} \\ M_{1,1,7} & M_{1,1,8} & M_{1,1,9} & M_{1,2,3} \\ \hline M_{2,1,1} & M_{2,1,2} & M_{2,1,3} & M_{2,2} \end{array}\right], \tag{7.78a}$$

$$\tilde{M} = \left[\begin{array}{ccc|c} M_{1,1,1} & \tilde{M}_{1,1,2} & M_{1,1,3} & M_{1,2,1} \\ \mathbf{0} & \tilde{M}_{1,1,5} & \mathbf{0} & \mathbf{0} \\ M_{1,1,7} & \tilde{M}_{1,1,8} & M_{1,1,9} & M_{1,2,3} \\ \hline M_{2,1,1} & \tilde{M}_{2,1,2} & M_{2,1,3} & M_{2,2} \end{array}\right], \tag{7.78b}$$

$$\check{M} = \left[\begin{array}{ccc|c} M_{1,1,1} & \mathbf{0} & M_{1,1,3} & M_{1,2,1} \\ \check{M}_{1,1,4} & \check{M}_{1,1,5} & \check{M}_{1,1,6} & \check{M}_{1,2,2} \\ M_{1,1,7} & \mathbf{0} & M_{1,1,9} & M_{1,2,3} \\ \hline M_{2,1,1} & \mathbf{0} & M_{2,1,3} & M_{2,2} \end{array}\right], \tag{7.78c}$$

$$\acute{M} = \left[\begin{array}{ccc|c} M_{1,1,1} & \mathbf{0} & M_{1,1,3} & M_{1,2,1} \\ \acute{M}_{1,1,4} & \acute{M}_{1,1,5} & \acute{M}_{1,1,6} & \acute{M}_{1,2,2} \\ M_{1,1,7} & \mathbf{0} & M_{1,1,9} & M_{1,2,3} \\ \hline M_{2,1,1} & \mathbf{0} & M_{2,1,3} & M_{2,2} \end{array}\right], \tag{7.78d}$$

admits the following relationship

$$\begin{aligned} &\mathcal{R}(\tilde{M}) = \mathcal{R}(M) = \mathcal{R}(\check{M}) = \mathcal{R}(\acute{M}) \\ &= -\left[\begin{array}{cc} M_{2,1,1} & M_{2,1,3} \end{array}\right] \left[\begin{array}{cc} M_{1,1,1} & M_{1,1,3} \\ M_{1,1,7} & M_{1,1,9} \end{array}\right]^{-1} \left[\begin{array}{c} M_{1,2,1} \\ M_{1,2,3} \end{array}\right] + M_{2,2}, \end{aligned} \tag{7.79}$$

with $\left[\begin{array}{cc} M_{1,1,1} & M_{1,1,3} \\ M_{1,1,7} & M_{1,1,9} \end{array}\right]$*,* $M_{1,1,5}$*,* $\tilde{M}_{1,1,5}$*,* $\check{M}_{1,1,5}$ *and* $\acute{M}_{1,1,5}$ *being invertible and* $M_{2,2} \in \mathbf{R}^{p\times q}$*.*

Proof. For simplicity, we only show the relationship about the matrix M of (7.78a), as the results for the other matrices is similar. Let

$$L = \begin{bmatrix} \mathbf{0} & I & \mathbf{0} & \mathbf{0} \\ I & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & I_p \end{bmatrix}, \quad R = \begin{bmatrix} \mathbf{0} & I & \mathbf{0} & \mathbf{0} \\ I & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & I_q \end{bmatrix}. \tag{7.80}$$

With attention Lemma 7.1 and

$$LMR = \left[\begin{array}{ccc|c} M_{1,1,5} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ M_{1,1,2} & M_{1,1,1} & M_{1,1,3} & M_{1,2,1} \\ M_{1,1,8} & M_{1,1,7} & M_{1,1,9} & M_{1,2,3} \\ \hline M_{2,1,2} & M_{2,1,1} & M_{2,1,3} & M_{2,2} \end{array}\right], \tag{7.81}$$

we have

$$\begin{aligned} \mathcal{R}(M) &= \mathcal{R}(LMR) \\ &= -\begin{bmatrix} M_{2,1,2} \; M_{2,1,1} \; M_{2,1,3} \end{bmatrix} \begin{bmatrix} M_{1,1,5} & \mathbf{0} & \mathbf{0} \\ M_{1,1,2} & M_{1,1,1} & M_{1,1,3} \\ M_{1,1,8} & M_{1,1,7} & M_{1,1,9} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{0} \\ M_{1,2,1} \\ M_{1,2,3} \end{bmatrix} + M_{2,2} \\ &= -\begin{bmatrix} M_{2,1,1} \; M_{2,1,3} \end{bmatrix} \begin{bmatrix} M_{1,1,1} & M_{1,1,3} \\ M_{1,1,7} & M_{1,1,9} \end{bmatrix}^{-1} \begin{bmatrix} M_{1,2,1} \\ M_{1,2,3} \end{bmatrix} + M_{2,2}. \end{aligned} \tag{7.82}$$

□

7.6.1 Proof of Theorem 7.2

Proof. Note that the given n-D Roesser state-space model $(A, B, C, D; \boldsymbol{r})$ is in the form as

$$A = \begin{bmatrix} X_{1,1} & \mathbf{0} & X_{1,3} & X_{1,4} & X_{1,5} \\ X_{2,1} & a_{\eta,\eta} & X_{2,3} & a_{\eta,\xi} & X_{2,5} \\ X_{3,1} & \mathbf{0} & X_{3,3} & X_{3,4} & X_{3,5} \\ \mathbf{0} & a_{\xi,\eta} & \mathbf{0} & a_{\xi,\xi} & \mathbf{0} \\ X_{5,1} & \mathbf{0} & X_{5,3} & X_{5,4} & X_{5,5} \end{bmatrix}, \quad B = \begin{bmatrix} X_{b1} \\ X_{b2} \\ X_{b3} \\ \mathbf{0} \\ X_{b5} \end{bmatrix}, \tag{7.83}$$
$$C = \begin{bmatrix} X_{c1} & \mathbf{0} & X_{c3} & X_{c4} & X_{c5} \end{bmatrix}.$$

where $X_{i,j}$, X_{bi}, X_{cj} are sub-block matrices of A, B, C, respectively, $i, j \in \{1, \ldots, 5\}$ such that the entries $a_{\eta,\eta}$ and $a_{\xi,\xi}$ are in the positions (η, η) and (ξ, ξ) of A, respectively. For the T defined in (7.27), we have

$$
T^{-1} = \begin{array}{c c}
\begin{array}{ccccc} & \downarrow \eta\text{th} & & \downarrow \xi\text{th} & \end{array} & \\
\begin{bmatrix} I & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 0 & \mathbf{0} & 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 1 & \mathbf{0} & -\gamma & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & I \end{bmatrix} & \begin{array}{l} \\ \leftarrow \eta\text{th} \\ \\ \leftarrow \xi\text{th} \\ \\ \end{array}
\end{array}. \tag{7.84}
$$

Then, the expected n-D Roesser state-space model $(\bar{A}, \bar{B}, \bar{C}, D; \bar{\boldsymbol{r}}) = (T^{-1}AT, T^{-1}B, CT, D; \boldsymbol{r})$ is in the form as

$$
\begin{aligned}
\bar{A} =& T^{-1}AT = \begin{bmatrix} X_{1,1} & X_{1,4} & X_{1,3} & \mathbf{0} & X_{1,5} \\ \mathbf{0} & a_{\eta,\eta} & \mathbf{0} & a_{\xi,\eta} & \mathbf{0} \\ X_{3,1} & X_{3,4} & X_{3,3} & \mathbf{0} & X_{3,5} \\ X_{2,1} & a_{\eta,\xi} & X_{2,3} & a_{\xi,\xi} & X_{25,} \\ X_{5,1} & X_{5,4} & X_{5,3} & \mathbf{0} & X_{5,5} \end{bmatrix}, \\
\bar{B} =& T^{-1}B = \begin{bmatrix} X_{b1} & \mathbf{0} & X_{b3} & X_{b2} & X_{b5} \end{bmatrix}^{\mathrm{T}}, \\
\bar{C} =& CT = \begin{bmatrix} X_{c1} & X_{c4} & X_{c3} & \mathbf{0} & X_{c5} \end{bmatrix}.
\end{aligned} \tag{7.85}
$$

Let

$$
Z = \operatorname{diag}\{z_1 I_{r_1}, \ldots, z_n I_{r_n}\}, \tag{7.86}
$$

and $z_1, \ldots, z_n$ denote the unit delay (backward-shift) operators. Compatible with A in (7.83), the Z can be partitioned as

$$
Z = \operatorname{diag}\{Z_1, Z_2, Z_3, Z_4, Z_5\}, \tag{7.87}
$$

where the size of $Z_1, \ldots, Z_5$ are respectively, $l_1, 1, l_3, 1, l_5$.

The column system matrix of n-D Roesser state-space model in (7.83) is

$$
M = \begin{bmatrix} I - AZ & B \\ -CZ & D \end{bmatrix} =
$$

$$
\begin{bmatrix} I_{l_1} - X_{1,1}Z_1 & \mathbf{0} & -X_{1,3}Z_3 & -X_{1,4}Z_4 & -X_{1,5}Z_5 & X_{b1} \\ -X_{2,1}Z_1 & 1 - a_{\eta,\eta}Z_2 & -X_{2,3}Z_3 & -a_{\eta,\xi}Z_4 & -X_{2,5}Z_5 & X_{b2} \\ -X_{3,1}Z_1 & \mathbf{0} & I_{l_3} - X_{3,3}Z_3 & -X_{3,4}Z_4 & -X_{3,5}Z_5 & X_{b3} \\ \mathbf{0} & -a_{\xi,\eta}Z_2 & \mathbf{0} & 1 - a_{\xi,\xi}Z_4 & \mathbf{0} & \mathbf{0} \\ -X_{5,1}Z_1 & \mathbf{0} & -X_{5,3}Z_3 & -X_{5,4}Z_4 & I_{l_5} - X_{5,5}Z_5 & X_{b5} \\ -X_{c1}Z_1 & \mathbf{0} & -X_{c3}Z_3 & -X_{c4}Z_4 & -X_{c5}Z_5 & D \end{bmatrix}. \tag{7.88}
$$

Let R_1 be

$$R_1 = \begin{array}{cc} \begin{array}{ccccccc} \phantom{I_{l_1}} & \downarrow\eta\text{th} & \phantom{\mathbf{0}} & \downarrow\xi\text{th} & \phantom{\mathbf{0}} & \phantom{\mathbf{0}} & \phantom{\mathbf{0}} \end{array} & \\ \begin{bmatrix} I_{l_1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 1 & \mathbf{0} & \mathbf{0} & 0 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & I_{l_3} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{a_{\xi,\eta}Z_2}{1-a_{\xi,\xi}Z_4} & \mathbf{0} & \mathbf{0} & 1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & I_{l_5} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & I_q \end{bmatrix} & \begin{array}{l} \\ \leftarrow \eta\text{th} \\ \\ \leftarrow \xi\text{th} \\ \\ \\ \end{array} \end{array}. \tag{7.89}$$

Then, multiplication of MR_1 gives

$$\begin{aligned} M_1 &\triangleq MR_1 \\ &= \begin{bmatrix} I_{l_1} - X_{1,1}Z_1 & \theta_1 & -X_{1,3}Z_3 & -X_{1,4}Z_4 & -X_{1,5}Z_5 & X_{b1} \\ -X_{2,1}Z_1 & 1+\theta_2 & -X_{2,3}Z_3 & -a_{\eta,\xi}Z_4 & -X_{2,5}Z_5 & X_{b2} \\ -X_{3,1}Z_1 & \theta_3 & I_{l_3} - X_{3,3}Z_3 & -X_{3,4}Z_4 & -X_{3,5}Z_5 & X_{b3} \\ \mathbf{0} & 0 & \mathbf{0} & 1-a_{\xi,\xi}Z_4 & \mathbf{0} & \mathbf{0} \\ -X_{5,1}Z_1 & \theta_5 & -X_{5,3}Z_3 & -X_{5,4}Z_4 & I_{l_5} - X_{5,5}Z_5 & X_{b5} \\ -X_{c1}Z_1 & \theta_6 & -X_{c3}Z_3 & -X_{c4}Z_4 & -X_{c5}Z_5 & D \end{bmatrix}, \end{aligned} \tag{7.90}$$

with $\theta_1 = -\frac{a_{\xi,\eta}Z_2}{1-a_{\xi,\xi}Z_4}X_{14}Z_4$, $\theta_2 = -a_{\eta,\eta}Z_2 - \frac{a_{\xi,\eta}Z_2}{1-a_{\xi,\xi}Z_4}a_{\eta,\xi}Z_4$, $\theta_3 = -\frac{a_{\xi,\eta}Z_2}{1-a_{\xi,\xi}Z_4}X_{34}Z_4$, $\theta_5 = -\frac{a_{\xi,\eta}Z_2}{1-a_{\xi,\xi}Z_4}X_{54}Z_4$ and $\theta_6 = -\frac{a_{\xi,\eta}Z_2}{1-a_{\xi,\xi}Z_4}X_{c4}Z_4$.

Define

$$\begin{aligned} M_2 &\triangleq \\ &\begin{bmatrix} I_{l_1} - X_{1,1}Z_1 & \theta_1 & -X_{1,3}Z_3 & -X_{1,4}Z_2 & -X_{1,5}Z_5 & X_{b1} \\ -X_{2,1}Z_1 & 1+\theta_2 & -X_{2,3}Z_3 & -a_{\eta,\xi}Z_2 & -X_{2,5}Z_5 & X_{b2} \\ -X_{3,1}Z_1 & \theta_3 & I_{l_3} - X_{3,3}Z_3 & -X_{3,4}Z_2 & -X_{3,5}Z_5 & X_{b3} \\ \mathbf{0} & 0 & \mathbf{0} & 1-a_{\xi,\xi}Z_4 & \mathbf{0} & \mathbf{0} \\ -X_{5,1}Z_1 & \theta_5 & -X_{5,3}Z_3 & -X_{5,4}Z_2 & I_{l_5} - X_{5,5}Z_5 & X_{b5} \\ -X_{c1}Z_1 & \theta_6 & -X_{c3}Z_3 & -X_{c4}Z_2 & -X_{c5}Z_5 & D \end{bmatrix}. \end{aligned} \tag{7.91}$$

By theorem 4.1, one can derived that

$$\mathcal{R}(M_1) = \mathcal{R}(M_2). \tag{7.92}$$

Let R_2 be

$$R_2 = \begin{array}{cl} \begin{array}{ccccccc} \phantom{I_{l_1}} & \downarrow \eta\text{th} & \phantom{\mathbf{0}} & \phantom{\mathbf{0}} & \downarrow \xi\text{th} & \phantom{\mathbf{0}} & \phantom{\mathbf{0}} \end{array} & \\ \begin{bmatrix} I_{l_1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 1 & \mathbf{0} & \mathbf{0} & 0 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & I_{l_3} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{-a_{\xi,\eta} Z_4}{1-a_{\xi,\xi} Z_4} & \mathbf{0} & \mathbf{0} & 1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & I_{l_5} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & I_q \end{bmatrix} & \begin{array}{l} \\ \leftarrow \eta\text{th} \\ \\ \leftarrow \xi\text{th} \\ \\ \\ \end{array} \end{array}. \tag{7.93}$$

Then, multiplication of $M_2 R_2$ gives

$$M_3 \triangleq M_2 R_2 =$$

$$\begin{bmatrix} I_{l_1} - X_{1,1} Z_1 & \mathbf{0} & -X_{1,3} Z_3 & -X_{1,4} Z_2 & -X_{1,5} Z_5 & X_{b1} \\ -X_{2,1} Z_1 & 1 - a_{\eta,\eta} Z_2 & -X_{2,3} Z_3 & -a_{\eta,\xi} Z_2 & -X_{2,5} Z_5 & X_{b2} \\ -X_{3,1} Z_1 & \mathbf{0} & I_{l_3} - X_{3,3} Z_3 & -X_{3,4} Z_2 & -X_{3,5} Z_5 & X_{b3} \\ \mathbf{0} & -a_{\xi,\eta} Z_4 & \mathbf{0} & 1 - a_{\xi,\xi} Z_4 & \mathbf{0} & \mathbf{0} \\ -X_{5,1} Z_1 & \mathbf{0} & -X_{5,3} Z_3 & -X_{5,4} Z_2 & I_{l_5} - X_{5,5} Z_5 & X_{b5} \\ -X_{c1} Z_1 & \mathbf{0} & -X_{c3} Z_3 & -X_{c4} Z_2 & -X_{c5} Z_5 & D \end{bmatrix}. \tag{7.94}$$

Define

$$R_3 = \begin{array}{cl} \begin{array}{ccccccc} \phantom{I_{l_1}} & \downarrow \eta\text{th} & \phantom{\mathbf{0}} & \phantom{\mathbf{0}} & \downarrow \xi\text{th} & \phantom{\mathbf{0}} & \phantom{\mathbf{0}} \end{array} & \\ \begin{bmatrix} I_{l_1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 1 & \mathbf{0} & \mathbf{0} & \alpha & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & I_{l_3} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & 1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & I_{l_5} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & I_q \end{bmatrix} & \begin{array}{l} \\ \leftarrow \eta\text{th} \\ \\ \leftarrow \xi\text{th'} \\ \\ \\ \end{array} \end{array} \tag{7.95a}$$

$$L_1 = \begin{array}{cl} \begin{array}{ccccccc} \phantom{I_{l_1}} & \downarrow \eta\text{th} & \phantom{\mathbf{0}} & \phantom{\mathbf{0}} & \downarrow \xi\text{th} & \phantom{\mathbf{0}} & \phantom{\mathbf{0}} \end{array} & \\ \begin{bmatrix} I_{l_1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 1 & \mathbf{0} & \mathbf{0} & -\alpha & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & I_{l_3} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & 1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & I_{l_5} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & I_p \end{bmatrix} & \begin{array}{l} \\ \leftarrow \eta\text{th} \\ \\ \leftarrow \xi\text{th'} \\ \\ \\ \end{array} \end{array} \tag{7.95b}$$

where $\alpha = \frac{a_{\eta,\eta} Z_2 - a_{\xi,\xi} Z_4}{a_{\xi,\eta} Z_4}$.

Then, multiplication of $L_1M_3R_3$ gives

$$
\begin{aligned}
M_4 &\triangleq L_1M_3R_3 \\
&= \begin{bmatrix}
I_{l_1}-X_{1,1}Z_1 & \mathbf{0} & -X_{1,3}Z_3 & -X_{1,4}Z_2 & -X_{1,5}Z_5 & X_{b1} \\
-X_{2,1}Z_1 & 1-a_{\xi,\xi}Z_4 & -X_{2,3}Z_3 & -a_{\eta,\xi}Z_2 & -X_{2,5}Z_5 & X_{b2} \\
-X_{3,1}Z_1 & \mathbf{0} & I_{l_3}-X_{3,3}Z_3 & -X_{3,4}Z_2 & -X_{3,5}Z_5 & X_{b3} \\
\mathbf{0} & -a_{\xi,\eta}Z_4 & \mathbf{0} & 1 & \mathbf{0} & \mathbf{0} \\
-X_{5,1}Z_1 & \mathbf{0} & -X_{5,3}Z_3 & -X_{5,4}Z_2 & I_{l_5}-X_{5,5}Z_5 & X_{b5} \\
-X_{c1}Z_1 & \mathbf{0} & -X_{c3}Z_3 & -X_{c4}Z_2 & -X_{c5}Z_5 & D
\end{bmatrix}.
\end{aligned} \tag{7.96}
$$

Let R_4 and L_4 be

$$
L_4 = \begin{array}{c@{}c}
\begin{array}{cccccc} & \downarrow \eta\text{th} & & \downarrow \xi\text{th} & & \end{array} & \\
\begin{bmatrix}
I_{l_1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & 0 & \mathbf{0} & 1 & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & I_{l_3} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & 1 & \mathbf{0} & 0 & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & I_{l_5} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & I_p
\end{bmatrix} &
\begin{array}{l} \\ \leftarrow \eta\text{th} \\ \\ \leftarrow \xi\text{th} \\ \\ \\ \end{array}
\end{array}, \tag{7.97a}
$$

$$
R_4 = \begin{array}{c@{}c}
\begin{array}{cccccc} & \downarrow \eta\text{th} & & \downarrow \xi\text{th,} & & \end{array} & \\
\begin{bmatrix}
I_{l_1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & 0 & \mathbf{0} & 1 & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & I_{l_3} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & 1 & \mathbf{0} & 0 & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & I_{l_5} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & I_q
\end{bmatrix} &
\begin{array}{l} \\ \leftarrow \eta\text{th} \\ \\ \leftarrow \xi\text{th} \\ \\ \\ \end{array}
\end{array}. \tag{7.97b}
$$

Then, we have

$$
\begin{aligned}
M_5 &\triangleq L_4M_4R_4 \\
&= \begin{bmatrix}
I_{l_1}-X_{1,1}Z_1 & -X_{1,4}Z_2 & -X_{1,3}Z_3 & \mathbf{0} & -X_{1,5}Z_5 & X_{b1} \\
\mathbf{0} & 1 & \mathbf{0} & -a_{\xi,\eta}Z_4 & \mathbf{0} & \mathbf{0} \\
-X_{3,1}Z_1 & -X_{3,4}Z_2 & I_{l_3}-X_{3,3}Z_3 & \mathbf{0} & -X_{3,5}Z_5 & X_{b3} \\
-X_{2,1}Z_1 & -a_{\eta,\xi}Z_2 & -X_{2,3}Z_3 & 1-a_{\xi,\xi}Z_4 & -X_{2,5}Z_5 & X_{b2} \\
-X_{5,1}Z_1 & -X_{5,4}Z_2 & -X_{5,3}Z_3 & \mathbf{0} & I_{l_5}-X_{5,5}Z_5 & X_{b5} \\
-X_{c1}Z_1 & -X_{c4}Z_2 & -X_{c3}Z_3 & \mathbf{0} & -X_{c5}Z_5 & D
\end{bmatrix}.
\end{aligned} \tag{7.98}
$$

From system matrix M_5, a new n-D Roesser model $\tilde{A}, \bar{B}, \bar{C}, D$ with order $(\bar{r}_1, \ldots, \bar{r}_n) =$

$(r_1, \ldots, r_n)$ can be obtained as

$$\bar{A} = \begin{bmatrix} X_{1,1} & X_{1,4} & X_{1,3} & \mathbf{0} & X_{1,5} \\ \mathbf{0} & 0 & \mathbf{0} & a_{\xi,\eta} & \mathbf{0} \\ X_{3,1} & X_{3,4} & X_{3,3} & \mathbf{0} & X_{3,5} \\ X_{2,1} & a_{\eta,\xi} & X_{2,3} & a_{\xi,\xi} & X_{2,5} \\ X_{5,1} & X_{5,4} & X_{5,3} & \mathbf{0} & X_{5,5} \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} X_{b1} \\ \mathbf{0} \\ X_{b3} \\ X_{b2} \\ X_{b5} \end{bmatrix},$$
$$\bar{C} = \begin{bmatrix} X_{c1} & X_{c4} & X_{c3} & \mathbf{0} & X_{c5} \end{bmatrix}, \tag{7.99}$$

which is equal to the expected n-D Roesser state-space model $(\bar{A}, \bar{B}, \bar{C}, D; \bar{\boldsymbol{r}})$ in (7.85). $\square$

7.7 Contribution Summary

In this paper, two types of transformations, i.e., non-structured similar transformations and general transformations, to obtain equivalent realizations have been established for an n-D Roesser state-space (state-space) model. It has been shown that applying non-structured similar transformations and general transformations to an n-D Roesser model can change noncommutative transfer matrix of this noncommutative Roesser model and hence the minimality in noncommutative Roesser model does not mean the minimality of normal commutative case. Based on those two types of transformations, a new exact order reduction approach for n-D Roesser model has been proposed. Moreover, it has been clarified that this exact order reduction approach can even reduce a minimal noncommutative Roesser model. Examples have been presented to illustrate the details as well as the effectiveness of the proposed approach.

Chapter 8

Conclusions and Future Works

This chapter summarizes the main results of this thesis and provides some possible future works.

8.1 Conclusions

This thesis has systematically studied the exact order reduction problem for state-space models of multidimensional systems. Theoretically, the relationship among exact order reduction for n-D state-space models, eigenvalues, eigenvectors, invariant subspaces has been fully established for the first time. It has turned out that the established connection for the n-D systems includes the results of the conventional 1-D case as a special case and also provides a basis for further development of multidimensional system theory. In application, the proposed approach could reduce a larger class of n-D state-space models than the existing reduction approaches. Specifically, the main contributions are drawn as follows.

1. **Eigenvalue Trim Approach to Exact Order Reduction for the Roesser Model** (Chapters 4)

   Inspired by the existing notion of trim or cotrim in [23], a new notion of eigenvalue trim and eigenvalue co-trim for n-D Roesser model has been introduced. Based on those new notions, a preliminary connection between the eigenvalues and the reducibility of the considered Roesser model has been established. Specifically, new sufficient conditions for reducibility and the corresponding order reduction algorithms for n-D Roesser model have been developed, which can achieve further order reduction than the existing approaches.

We would like to remark that the reduction conditions of the eigenvalue trim approach only focused on the eigenvalues of one sub-matrix corresponding one variable and the task of providing a full exploration on the reducibility of n-D Roesser model by simultaneously taking into account the eigenvalues of all the blocks w.r.t. all the variables will be accomplished by the following approach.

2. **Common Eigenvector Approach to Exact Order Reduction for State-space Models of Multidimensional Systems** (Chapters 5)

   The notion of constrained common eigenvector has been introduced, which can simultaneously take into account the eigenvalues of all the states-matrices of the F-M model and the eigenvalues of all the blocks in the system matrix of the Roesser model. Based on this constrained common eigenvector, sufficient reducibility conditions the n-D F-M model and the n-D Roesser model have been developed, which can be viewed as a kind of generalization of PBH tests for the exact reducibility of n-D state-space models. A Gröbner basis approach has also been proposed to compute such a constrained common eigenvector. Moreover, a generalization to the state delay case has been given to show this method more applicable.

3. **Common Invariant Subspace Approach to Exact Order Reduction for State-space Models of Multidimensional Systems** (Chapter 6)

   A common invariant subspace approach has been established to state-space models of multidimensional systems. Specifically, new sufficient reducibility conditions based on common invariant subspaces have been developed for the F-M model and Roesser model, respectively. Moreover, it has been shown that these conditions are necessary reducibility in the noncommutative setting. It also shows that the common invariant subspace approach includes the common eigenvector approach as a special case. Based on these new reducibility conditions, new constructive reduction procedures are given for the F-M model and the Roesser model, respectively.

4. **Further Exact Order Reduction** (Chapter 7)

   The exact order reduction for Roesser model has been further studied based on equivalence relationships. In particular, two types of transformations obtain equivalent realizations have been established for the Roesser model. It has been shown that applying those two equivalent transformations to a minimal n-D Roesser model in the noncommutative setting could change the noncommutative transfer matrix of this n-D Roesser model and then the transformed n-D Roesser model could be reduced again by applying the common invariant subspace approach. Based on this fact, a novel reduction procedure has been presented, which repeatedly applies the common invariant subspace approach to generate minimal Roesser model realization in the noncommutative setting and the two equivalent transformations to obtain another Roesser model with different noncommutative transfer function matrices, such that an n-D Roesser model with order as low as possible can be obtained.

8.2 Future Work

Some directions for future research are as follows.

- In Chapter 7, it has been shown that the equivalence transformations play an important role in the reduction process, and two basic types of equivalence transformations have been established for the Roesser model. To fully study the exact reduction problem for the Roesser model, more equivalence transformations have to be further studied. Furthermore, to fully study the exact reduction problem of the F-M model, it is necessary to extend the equivalence transformations developed for the Roesser model to the F-M model.

- Problems connected with ideals generated by finite sets of multivariate polynomials occur as mathematical sub-problems in various branches of systems theory [90]. The method of Gröbner bases is a technique that provides algorithmic solutions to a variety of such problems. One of representative applications of Gröbner bases is that it can find solutions to systems of polynomial equations. On the other hand, multidimensional systems are characterized by n ($n > 1$) independent variables which can be regarded as a special ring of the multivariate ring. It would be very interesting to investigate the exact order problem of multidimensional systems based on Gröbner bases.

- This thesis only focuses on the exact order reduction problem for the multidimensional systems. The generalization of the proposed methods to the approximate order reduction would also be very interesting.

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