Short Report

Duality of Nonhomogeneous Semi-Fibonacci Programming

— Inequality Approach —

Yutaka Kimura[1] and Seiichi Iwamoto[2]

[1]Faculty of Systems Science and Technology, Akita Prefectural University
[2]Professor Emeritus, Kyushu University

We consider a pair of a minimization (primal) problem and a maximization (dual) problem. The objective functions are quadratic and the constraints are linear. Both constraints are called *nonhomogeneously semi-Fibonacci*. Thus both problems are called *semi-Fibonacci programming* under a *nonhomogeneous constraint*. In this paper, we discuss how to derive the dual problem from the primal one. Thus, we present an inequality method. The inequality method applies the Arithmetic-mean/Geometric-mean inequality. Moreover, it is shown that the pair has *Fibonacci identical duality* (FID) in the following sense. (i) (Duality) Both problems are dual to each other. (ii) (Identical) Both problems have an identical optimal solution (point and value). (iii) (Fibonacci) The optimal solution is Fibonacci in the sense. The identical optimum point is linear, while the identical optimal value is quadratic. Thus the pair of problems satisfies FID.

Keywords： Optimizatoin, quadratic programming, Fibonacci sequence, dualization

Introduction

In this paper, we consider a pair of a $2n$-variable conditional minimization problem

$$
(\mathrm{P})\quad
\begin{array}{ll}
\text{minimize} & y_1^2 + y_2^2 + \cdots + y_{2n}^2 \\
& \quad -2(b_1y_1 + b_3y_3 + \cdots + b_{2n-1}y_{2n-1}) \\
\text{subject to} & (1)\ \ y_1 + y_2 - y_3 = b_2 \\
& (2)\ \ y_3 + y_4 - y_5 = b_4 \\
& \qquad\qquad \vdots \\
& (\mathrm{n}-1)\ \ y_{2n-3} + y_{2n-2} - y_{2n-1} = b_{2n-2} \\
& (\mathrm{n})\ \ y_{2n-1} + y_{2n} = b_{2n} \\
& (\mathrm{n}+1)\ \ y \in R^{2n}
\end{array}
$$

and a $2n$-variable problem

$$
(\mathrm{D})\quad
\begin{array}{ll}
\text{Maximize} & -(\mu_1^2 + \mu_2^2 + \cdots + \mu_{2n}^2) \\
& \quad +2(b_2\mu_2 + b_4\mu_4 + \cdots + b_{2n}\mu_{2n}) \\
\text{subject to} & (1)'\ \ \mu_1 - \mu_2 = b_1 \\
& (2)'\ \ \mu_2 + \mu_3 - \mu_4 = b_3 \\
& (3)'\ \ \mu_4 + \mu_5 - \mu_6 = b_5 \\
& \qquad\qquad \vdots \\
& (\mathrm{n})'\ \ \mu_{2n-2} + \mu_{2n-1} - \mu_{2n} = b_{2n-1} \\
& (\mathrm{n}+1)'\ \ \mu \in R^{2n}
\end{array}
$$

where $b = (b_1, b_2, \ldots, b_{2n}) \in R^{2n}$ is a constant.

The objective functions are quadratic, while the constraints (1)∼(n) and (1)′∼(n)′ are linear. Both constraints are called *nonhomogeneously semi-Fibonacci*.

Thus both problems are called *semi-Fibonacci programming* under a *nonhomogeneous constraint*. The constraint with $b_1 = b_2 = \cdots = b_{2n} = 0$ is called *homogeneous*. The semi-Fibonacci programming is a quadratic programming.

This expression suggests how to express the $2n$-variable pair (P), (D). It turns out that the pair has an identical optimal solution (point and value), which is characterized by the first $(2n+1)$ Fibonacci numbers (Table 1):

$$F_1,\ F_2,\ \ldots,\ F_{2n},\ F_{2n+1}.$$

The *Fibonacci sequence* $\{F_n\}$ is defined as the solution to the second-order linear difference equation,

$$x_{n+2} - x_{n+1} - x_n = 0, \qquad x_1 = 1,\ x_0 = 0. \tag{1}$$

Table 1　Fibonacci sequence $\{F_n\}$

n	0	1	2	3	4	5	6	7	8	9	10	11
F_n	0	1	1	2	3	5	8	13	21	34	55	89

Corresponding author: Yutaka Kimura 84-4, Tsuchiya-Ebinokuchi, Yurihonjo, Akita 015-0055, Department of Man- agement Science and Engineering, Faculty of Systems Science and Technology, Akita Prefectural University, E-mail:yutaka@akita-pu.ac.jp

Dualization – Inequality Method –

First we discuss a pair of an 8-variable conditional minimization problem

$$(\mathrm{P})' \quad \begin{array}{ll} \text{minimize} & y_1^2+y_2^2+y_3^2+y_4^2+y_5^2+y_6^2 \\ & +y_7^2+y_8^2-2(b_1y_1+b_3y_3+b_5y_5+b_7y_7) \\ \text{subject to} & \text{(i)}\ \ y_1+y_2-y_3 = b_2 \\ & \text{(ii)}\ \ y_3+y_4-y_5 = b_4 \\ & \text{(iii)}\ \ y_5+y_6-y_7 = b_6 \\ & \text{(iv)}\ \ y_7+y_8 = b_8 \\ & \text{(v)}\ \ y \in R^8 \end{array}$$

and an 8-variable conditional maximization problem

$$(\mathrm{D})' \quad \begin{array}{ll} \text{Maximize} & -(\mu_1^2+\mu_2^2+\mu_3^2+\mu_4^2+\mu_5^2+\mu_6^2+\mu_7^2+\mu_8^2) \\ & +2(b_2\mu_2+b_4\mu_4+b_6\mu_6+b_8\mu_8) \\ \text{subject to} & \text{(i)}'\ \ \mu_1-\mu_2 = b_1 \\ & \text{(ii)}'\ \ \mu_2+\mu_3-\mu_4 = b_3 \\ & \text{(iii)}'\ \ \mu_4+\mu_5-\mu_6 = b_5 \\ & \text{(iv)}'\ \ \mu_6+\mu_7-\mu_8 = b_7 \\ & \text{(v)}'\ \ \mu \in R^8 \end{array}$$

where $b = (b_1, b_2, \ldots, b_8) \in R^8$ is a constant.

Theorem 1 (Duality Theorem)

(i) (Weak Duality) *It holds that $g(\mu) \le f(y)$ for any pair of feasible solutions (y, μ).*

(ii) (Strong Duality) *There exists a pair of feasible solutions $(\hat{y}, \mu^*)$ satisfying $f(\hat{y}) = g(\mu^*)$.*

(iii) (Optimal Solution) *The solution $\hat{y}$ is an optimal solution for* (P)′*, while the solution μ^* is an optimal solution for* (D)′.

Note that (ii) implies (iii). In the following we show (i) and (ii).

Lemma 1 (Complementarity) *It holds that*

$$\sum_{k=1}^{8} y_k\mu_k = (b_1y_1+b_3y_3+b_5y_5+b_7y_7) +(b_2\mu_2+b_4\mu_4+b_6\mu_6+b_8\mu_8) \quad (2)$$

under the constraints (i)∼(iv) *and* (i)′∼(iv)′*:*

$$\begin{array}{rclrcl} y_1+y_2-y_3 &=& b_2 & \mu_1-\mu_2 &=& b_1 \\ y_3+y_4-y_5 &=& b_4 & \mu_2+\mu_3-\mu_4 &=& b_3 \\ y_5+y_6-y_7 &=& b_6 & \mu_4+\mu_5-\mu_6 &=& b_5 \\ y_7+y_8 &=& b_8 & \mu_6+\mu_7-\mu_8 &=& b_7. \end{array}$$

Proof.

$$\begin{aligned} \sum_{k=1}^{8} y_k\mu_k &= y_1(\mu_2+b_1)+(y_3-y_1+b_2)\mu_2 \\ &\quad +y_3(\mu_4-\mu_2+b_3)+(y_5-y_3+b_4)\mu_4 \\ &\quad +y_5(\mu_6-\mu_4+b_5)+(y_7-y_5+b_6)\mu_6 \\ &\quad +y_7(\mu_8-\mu_6+b_7)+(b_8-y_7)\mu_8 \\ &= (b_1y_1+b_3y_3+b_5y_5+b_7y_7) \\ &\quad +(b_2\mu_2+b_4\mu_4+b_6\mu_6+b_8\mu_8). \end{aligned}$$

□

Lemma 2

(i) (Inequality) *It holds that $g(\mu) \le f(y)$ for any feasible solutions y, μ.*

(ii) (Equality) *The sign of equality holds if and only if $y = \mu$.*

(iii) (Linearity) *Furthermore it holds that*

$$\begin{aligned} f(y) &= -(b_1y_1+b_3y_3+b_5y_5+b_7y_7) \\ &\quad +(b_2y_2+b_4y_4+b_6y_6+b_8y_8) \end{aligned} \quad (3)$$

$$\begin{aligned} g(\mu) &= -(b_1\mu_1+b_3\mu_3+b_5\mu_5+b_7\mu_7) \\ &\quad +(b_2\mu_2+b_4\mu_4+b_6\mu_6+b_8\mu_8). \end{aligned} \quad (4)$$

Proof. (i) Our approach is based upon an elementary inequality with equality condition

$$2x\lambda \le x^2+\lambda^2 \quad \text{on } R^2\,; \ x=\lambda.$$

Let y and μ satisfy the constraints (i)∼(iv) and (i)′∼(iv)′, respectively. Then by applying the inequality 8 times and summing over $k = 1, 2, \ldots, 8$, we have the inequality with equality condition:

$$2\sum_{k=1}^{8} y_k\mu_k \le \sum_{k=1}^{8} y_k^2 + \sum_{k=1}^{8} \mu_k^2 \,; y_k=\mu_k \quad 1 \le k \le 8. \quad (5)$$

Lemma 1 implies that

$$\sum_{k=1}^{8} y_k\mu_k = (b_1y_1+b_3y_3+b_5y_5+b_7y_7) +(b_2\mu_2+b_4\mu_4+b_6\mu_6+b_8\mu_8). \quad (6)$$

From (5) and (6), we have an inequality

$$\begin{aligned} &-\sum_{k=1}^{8} \mu_k^2 + 2(b_2\mu_2+b_4\mu_4+b_6\mu_6+b_8\mu_8) \\ &\quad \le \sum_{k=1}^{8} y_k^2 - 2(b_1y_1+b_3y_3+b_5y_5+b_7y_7). \end{aligned} \quad (7)$$

Hence it holds that $g(\mu) \le f(y)$ for any feasible y and μ.

(ii) The sign of equality holds if and only if

$$\text{(e)} \quad y_k = \mu_k \quad 1 \le k \le 8.$$

Thus (P)′ and (D)′ are dual to each other.

(iii) Furthermore, the complementarity (2) with $y_k = \mu_k \ \ 1 \le k \le 8$ yields

$$\begin{aligned} \sum_{k=1}^{8} y_k^2 &= \sum_{k=1}^{8} \mu_k^2 \\ &= (b_1y_1+b_3y_3+b_5y_5+b_7y_7) \\ &\quad +(b_2\mu_2+b_4\mu_4+b_6\mu_6+b_8\mu_8). \end{aligned} \quad (8)$$

Thus we obtain

$$\begin{aligned} f(y) &= y_1^2 + \cdots + y_8^2 - 2(b_1y_1 + b_3y_3 + b_5y_5 + b_7y_7) \\ &= (b_1y_1 + b_3y_3 + b_5y_5 + b_7y_7) \\ &\qquad +(b_2\mu_2 + b_4\mu_4 + b_6\mu_6 + b_8\mu_8) \\ &\qquad -2(b_1y_1 + b_3y_3 + b_5y_5 + b_7y_7) \\ &= -(b_1y_1 + b_3y_3 + b_5y_5 + b_7y_7) \\ &\qquad +(b_2y_2 + b_4y_4 + b_6y_6 + b_8y_8) \end{aligned}$$

$$\begin{aligned} g(\mu) &= -(\mu_1^2 + \cdots + \mu_8^2) + 2(b_2\mu_2 + b_4\mu_4 + b_6\mu_6 + b_8\mu_8) \\ &= -(b_1y_1 + b_3y_3 + b_5y_5 + b_7y_7) \\ &\qquad -(b_2\mu_2 + b_4\mu_4 + b_6\mu_6 + b_8\mu_8) \\ &\qquad +2(b_2\mu_2 + b_4\mu_4 + b_6\mu_6 + b_8\mu_8) \\ &= -(b_1\mu_1 + b_3\mu_3 + b_5\mu_5 + b_7\mu_7) \\ &\qquad +(b_2\mu_2 + b_4\mu_4 + b_6\mu_6 + b_8\mu_8). \end{aligned}$$

□

Thus it holds that

$$g(\mu) \le f(y)$$

for any feasible pair (y, μ). The equality $g(\mu) = f(y)$ holds iff (i)〜(iv), (e), (i)′〜(iv)′ holds :

$$(\mathrm{EC})' \quad \begin{aligned} y_1 + y_2 - y_3 &= b_2 & \mu_1 - \mu_2 &= b_1 \\ y_3 + y_4 - y_5 &= b_4 & \mu_2 + \mu_3 - \mu_4 &= b_3 \\ y_5 + y_6 - y_7 &= b_6 & \mu_4 + \mu_5 - \mu_6 &= b_5 \\ y_7 + y_8 &= b_8 & \mu_6 + \mu_7 - \mu_8 &= b_7 \end{aligned}$$
$$y_k = \mu_k \quad 1 \le k \le 8.$$

This is a system of 16 linear equations in 16 variables, which is equivalent to two (identical) systems of 8 equations in 8 variables

$$\begin{aligned} y_1 - y_2 &= b_1 & y_1 + y_2 - y_3 &= b_2 \\ y_2 + y_3 - y_4 &= b_3 & y_3 + y_4 - y_5 &= b_4 \\ y_4 + y_5 - y_6 &= b_5 & y_5 + y_6 - y_7 &= b_6 \\ y_6 + y_7 - y_8 &= b_7 & y_7 + y_8 &= b_8 \end{aligned}$$
$$y_k = \mu_k \quad 1 \le k \le 8.$$

From Lemma 3, (EC)′ has a unique identical solution

$$(\hat{y}_1,\ \hat{y}_2,\ \ldots,\ \hat{y}_8) = (\mu_1^*,\ \mu_2^*,\ \ldots,\ \mu_8^*);$$

$$\begin{aligned} \hat{y}_1 &= \mu_1^* = \frac{1}{34}(21b_1 + 13b_2 + 8b_3 + 5b_4 + 3b_5 + 2b_6 + b_7 + b_8) \\ \hat{y}_2 &= \mu_2^* = \frac{1}{34}(-13b_1 + 13b_2 + 8b_3 + 5b_4 + 3b_5 + 2b_6 + b_7 + b_8) \\ \hat{y}_3 &= \mu_3^* = \frac{1}{34}(8b_1 - 8b_2 + 16b_3 + 10b_4 + 6b_5 + 4b_6 + 2b_7 + 2b_8) \\ \hat{y}_4 &= \mu_4^* = \frac{1}{34}(-5b_1 + 5b_2 - 10b_3 + 15b_4 + 9b_5 + 6b_6 + 3b_7 + 3b_8) \\ \hat{y}_5 &= \mu_5^* = \frac{1}{34}(3b_1 - 3b_2 + 6b_3 - 9b_4 + 15b_5 + 10b_6 + 5b_7 + 5b_8) \\ \hat{y}_6 &= \mu_6^* = \frac{1}{34}(-2b_1 + 2b_2 - 4b_3 + 6b_4 - 10b_5 + 16b_6 + 8b_7 + 8b_8) \\ \hat{y}_7 &= \mu_7^* = \frac{1}{34}(b_1 - b_2 + 2b_3 - 3b_4 + 5b_5 - 8b_6 + 13b_7 + 13b_8) \\ \hat{y}_8 &= \mu_8^* = \frac{1}{34}(-b_1 + b_2 - 2b_3 + 3b_4 - 5b_5 + 8b_6 - 13b_7 + 21b_8). \end{aligned}$$

Further from (3), (4) it turns out that a minimum value

$$m' = \sum_{k=1}^{8} \hat{y}_k^2 - 2(b_1\hat{y}_1 + b_3\hat{y}_3 + b_5\hat{y}_5 + b_7\hat{y}_7)$$

is linear with respect to the minimum point $\hat{y} = (\hat{y}_1,\ \hat{y}_2,\ \ldots,\ \hat{y}_8)$:

$$\begin{aligned} m' &= -(b_1\hat{y}_1 + b_3\hat{y}_3 + b_5\hat{y}_5 + b_7\hat{y}_7) \\ &\qquad +(b_2\hat{y}_2 + b_4\hat{y}_4 + b_6\hat{y}_6 + b_8\hat{y}_8) \end{aligned}$$

and the *same* maximum value

$$M' = -\sum_{k=1}^{8} \mu_k^{*2} + 2(b_2\mu_2^* + b_4\mu_4^* + b_6\mu_6^* + b_8\mu_8^*)$$

is linear with respect to the maximum point $\mu^* = (\mu_1^*,\ \mu_2^*,\ \ldots,\ \mu_8^*)$:

$$\begin{aligned} M' &= -(b_1\mu_1^* + b_3\mu_3^* + b_5\mu_5^* + b_7\mu_7^*) \\ &\qquad +(b_2\mu_2^* + b_4\mu_4^* + b_6\mu_6^* + b_8\mu_8^*). \end{aligned}$$

Lemma 3 (Fibonacci solution) *The system of 8 linear equations in 8 variables*

$$(\mathrm{NF})' \quad \begin{aligned} y_1 - y_2 &= b_1 & y_1 + y_2 - y_3 &= b_2 \\ y_2 + y_3 - y_4 &= b_3 & y_3 + y_4 - y_5 &= b_4 \\ y_4 + y_5 - y_6 &= b_5 & y_5 + y_6 - y_7 &= b_6 \\ y_6 + y_7 - y_8 &= b_7 & y_7 + y_8 &= b_8 \end{aligned}$$

has a unique solution $y = A^{-1}b$:

$$y = \frac{1}{34}\begin{pmatrix} 21b_1 + 13b_2 + 8b_3 + 5b_4 + 3b_5 + 2b_6 + b_7 + b_8 \\ -13b_1 + 13b_2 + 8b_3 + 5b_4 + 3b_5 + 2b_6 + b_7 + b_8 \\ 8b_1 - 8b_2 + 16b_3 + 10b_4 + 6b_5 + 4b_6 + 2b_7 + 2b_8 \\ -5b_1 + 5b_2 - 10b_3 + 15b_4 + 9b_5 + 6b_6 + 3b_7 + 3b_8 \\ 3b_1 - 3b_2 + 6b_3 - 9b_4 + 15b_5 + 10b_6 + 5b_7 + 5b_8 \\ -2b_1 + 2b_2 - 4b_3 + 6b_4 - 10b_5 + 16b_6 + 8b_7 + 8b_8 \\ b_1 - b_2 + 2b_3 - 3b_4 + 5b_5 - 8b_6 + 13b_7 + 13b_8 \\ -b_1 + b_2 - 2b_3 + 3b_4 - 5b_5 + 8b_6 - 13b_7 + 21b_8 \end{pmatrix}, \quad (9)$$

where

$$A = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \quad (10)$$

and A^{-1} is the inverse matrix of A.

□

The minimum point $\hat{y}$ and the maximum point μ^* are expressed as a linear form :

$$\hat{y} = \mu^* = A^{-1}b$$

where A^{-1} is

$$\frac{1}{F_9}\begin{pmatrix} F_8 & F_7 & F_6 & F_5 & F_4 & F_3 & F_2 & F_1 \\ -F_7 & F_7F_2 & F_6F_2 & F_5F_2 & F_4F_2 & F_3F_2 & F_2F_2 & F_2 \\ F_6 & -F_6F_2 & F_6F_3 & F_5F_3 & F_4F_3 & F_3F_3 & F_2F_3 & F_3 \\ -F_5 & F_5F_2 & -F_5F_3 & F_5F_4 & F_4F_4 & F_3F_4 & F_2F_4 & F_4 \\ F_4 & -F_4F_2 & F_4F_3 & -F_4F_4 & F_4F_5 & F_3F_5 & F_2F_5 & F_5 \\ -F_3 & F_3F_2 & -F_3F_3 & F_3F_4 & -F_3F_5 & F_3F_6 & F_2F_6 & F_6 \\ F_2 & -F_2F_2 & F_2F_3 & -F_2F_4 & F_2F_5 & -F_2F_6 & F_2F_7 & F_7 \\ -F_1 & F_2 & -F_3 & F_4 & -F_5 & F_6 & -F_7 & F_8 \end{pmatrix}.$$

The minimum value m' and the maximum value M' are a quadratic form :

$$m' \ = \ M' = (b, Bb)$$

where B is

$$\frac{1}{F_9}\begin{pmatrix} -F_8 & -F_7 & -F_6 & -F_5 & -F_4 & -F_3 & -F_2 & -F_1 \\ -F_7 & F_7F_2 & F_6F_2 & F_5F_2 & F_4F_2 & F_3F_2 & F_2F_2 & F_2 \\ -F_6 & F_6F_2 & -F_6F_3 & -F_5F_3 & -F_4F_3 & -F_3F_3 & -F_2F_3 & -F_3 \\ -F_5 & F_5F_2 & -F_5F_3 & F_5F_4 & F_4F_4 & F_3F_4 & F_2F_4 & F_4 \\ -F_4 & F_4F_2 & -F_4F_3 & F_4F_4 & -F_4F_5 & -F_3F_5 & -F_2F_5 & -F_5 \\ -F_3 & F_3F_2 & -F_3F_3 & F_3F_4 & -F_3F_5 & F_3F_6 & F_2F_6 & F_6 \\ -F_2 & F_2F_2 & -F_2F_3 & F_2F_4 & -F_2F_5 & F_2F_6 & -F_2F_7 & -F_7 \\ -F_1 & F_2 & -F_3 & F_4 & -F_5 & F_6 & -F_7 & F_8 \end{pmatrix}.$$

Both problems have an *identical optimal solution* characterized by the first nine Fibonacci numbers

$$F_1, \ F_2, \ \ldots, \ F_9.$$

Thus $(\mathrm{P})'$ and $(\mathrm{D})'$ satisfy *Fibonacci identical duality* (FID):

1. (Duality) Both problems are dual to each other.
2. (Identical) Both problems have an identical optimal solution (point and value): $(\hat{y}, m') = (\mu^*, M')$.
3. (Fibonacci) The optimal solution is Fibonacci in the following sense. The identical optimum point $\hat{y} = \mu^* = A^{-1}b$ is linear, while the identical optimal value $m' = M' = (b, Bb)$ is quadratic, where $B = JA^{-1}$;

   $$J = \mathrm{diag}(-1, 1, -1, 1, -1, 1, -1, 1),$$

   and the matrix A is same as (10).

Note that the inverse matrix $A^{-1} = \dfrac{1}{F_9}(a_{ij})$ consists of the four frames (first row, last row, first column and last column):

$$\begin{aligned} &(\ a_{11}, \ a_{12}, \ a_{13}, \ a_{14}, \ a_{15}, \ a_{16}, \ a_{17}, \ a_{18} \ ) \\ = \ &(\ F_8, \ F_7, \ F_6, \ F_5, \ F_4, \ F_3, \ F_2, \ F_1 \ ) \\ &(\ a_{81}, \ a_{82}, \ a_{83}, \ a_{84}, \ a_{85}, \ a_{86}, \ a_{87}, \ a_{88} \ ) \\ = \ &(\ -F_1, \ F_2, \ -F_3, \ F_4, \ -F_5, \ F_6, \ -F_7, \ F_8 \ ), \end{aligned}$$

$$\begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \\ a_{41} \\ a_{51} \\ a_{61} \\ a_{71} \\ a_{81} \end{pmatrix} = \begin{pmatrix} F_8 \\ -F_7 \\ F_6 \\ -F_5 \\ F_4 \\ -F_3 \\ F_2 \\ -F_1 \end{pmatrix}, \quad \begin{pmatrix} a_{18} \\ a_{28} \\ a_{38} \\ a_{48} \\ a_{58} \\ a_{68} \\ a_{78} \\ a_{88} \end{pmatrix} = \begin{pmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \\ F_7 \\ F_8 \end{pmatrix}.$$

That is,

$$\begin{aligned} a_{1j} \ &= \ F_{9-j}, \quad a_{8j} \ = \ (-1)^j F_j \\ a_{i1} \ &= \ (-1)^{i-1} F_{9-i}, \quad a_{i8} \ = \ F_i. \end{aligned}$$

These elements are called *outer*. The other elements are called *inner*. All the inner elements are determined by two outer elements through multiplication:

$$\begin{aligned} a_{ij} \ &= \ \begin{cases} a_{i8}a_{1j} & i \le j \\ -a_{i1}a_{8j} & i > j \end{cases} \\ &= \ \begin{cases} F_iF_{9-j} & i \le j \\ (-1)^{i+j}F_{9-i}F_j & i > j. \end{cases} \end{aligned}$$

$2n$-variable Pair

Let us consider a pair of a $2n$-variable problem

$$(\mathrm{P}) \quad \begin{array}{ll} \text{minimize} & y_1^2 + y_2^2 + \cdots + y_{2n}^2 \\ & \quad -2(b_1y_1 + b_3y_3 + \cdots + b_{2n-1}y_{2n-1}) \\ \text{subjectto} & (1) \ \ y_1 + y_2 - y_3 \ = \ b_2 \\ & (2) \ \ y_3 + y_4 - y_5 \ = \ b_4 \\ & \qquad \vdots \\ & (\mathrm{n}-1) \ \ y_{2n-3} + y_{2n-2} - y_{2n-1} \ = \ b_{2n-2} \\ & (\mathrm{n}) \ \ y_{2n-1} + y_{2n} \ = \ b_{2n} \\ & (\mathrm{n}+1) \ \ y \in R^{2n} \end{array}$$

and a $2n$-variable problem

$$(\mathrm{D}) \quad \begin{array}{ll} \text{Maximize} & -(\mu_1^2 + \mu_2^2 + \cdots + \mu_{2n}^2) \\ & \quad +2(b_2\mu_2 + b_4\mu_4 + \cdots + b_{2n}\mu_{2n}) \\ \text{subjectto} & (1)' \ \ \mu_1 - \mu_2 \ = \ b_1 \\ & (2)' \ \ \mu_2 + \mu_3 - \mu_4 \ = \ b_3 \\ & (3)' \ \ \mu_4 + \mu_5 - \mu_6 \ = \ b_5 \\ & \qquad \vdots \\ & (\mathrm{n})' \ \ \mu_{2n-2} + \mu_{2n-1} - \mu_{2n} \ = \ b_{2n-1} \\ & (\mathrm{n}+1)' \ \ \mu \in R^{2n} \end{array}$$

where $b = (b_1, b_2, \ldots, b_{2n}) \in R^{2n}$ is a constant. Let us denote the objective functions by f and g :

$$\begin{aligned} f(y) \ &= \ y_1^2 + y_2^2 + \cdots + y_{2n}^2 \\ &\qquad -2(b_1y_1 + b_3y_3 + \cdots + b_{2n-1}y_{2n-1}) \\ g(\mu) \ &= \ -(\mu_1^2 + \mu_2^2 + \cdots + \mu_{2n}^2) \\ &\qquad +2(b_2\mu_2 + b_4\mu_4 + \cdots + b_{2n}\mu_{2n}). \end{aligned}$$

Theorem 2 (Duality Theorem)
(i) (Weak Duality) *It holds that $g(\mu) \le f(y)$ for any pair of feasible solutions (y, μ).*
(ii) (Strong Duality) *There exists a pair of feasible solutions $(\hat{y}, \mu^*)$ satisfying $f(\hat{y}) = g(\mu^*)$.*
(iii) (Optimal Solution) *The solution $\hat{y}$ is optimal for* (P), *while μ^* is optimal for* (D).

Lemma 4 (Equality) *It holds that*

$$\begin{aligned} \sum_{k=1}^{2n} y_k\mu_k \ &= \ (b_1y_1 + b_3y_3 + \cdots + b_{2n-1}y_{2n-1}) \\ &\qquad +(b_2\mu_2 + b_4\mu_4 + \cdots + b_{2n}\mu_{2n}) \quad (11) \end{aligned}$$

under the constraints (1)〜(n) *and* (1)′〜(n)′:

$$
\begin{aligned}
y_1 + y_2 - y_3 &= b_2\\
y_3 + y_4 - y_5 &= b_4\\
y_5 + y_6 - y_7 &= b_6\\
&\vdots\\
y_{2n-3} + y_{2n-2} - y_{2n-1} &= b_{2n-2}\\
y_{2n-1} + y_{2n} &= b_{2n}\\
\mu_1 - \mu_2 &= b_1\\
\mu_2 + \mu_3 - \mu_4 &= b_3\\
\mu_4 + \mu_5 - \mu_6 &= b_5\\
&\vdots\\
\mu_{2n-4} + \mu_{2n-3} - \mu_{2n-2} &= b_{2n-3}\\
\mu_{2n-2} + \mu_{2n-1} - \mu_{2n} &= b_{2n-1}.
\end{aligned}
$$

Lemma 5

(i) (Inequality) *It holds that* $g(\mu) \le f(y)$ *for any feasible solutions* y, μ.

(ii) (Equality) *The sign of equality holds if and only iff* $y = \mu$.

(iii) (Linearity) *Furthermore it holds that*

$$
\begin{aligned}
f(y) &= -(b_1y_1 + b_3y_3 + \cdots + b_{2n-1}y_{2n-1})\\
&\quad +(b_2y_2 + b_4y_4 + \cdots + b_{2n}y_{2n}) \qquad (12)
\end{aligned}
$$

$$
\begin{aligned}
g(\mu) &= -(b_1\mu_1 + b_3\mu_3 + \cdots + b_{2n-1}\mu_{2n-1})\\
&\quad +(b_2\mu_2 + b_4\mu_4 + \cdots + b_{2n}\mu_{2n}). \qquad (13)
\end{aligned}
$$

Thus it holds that

$$g(\mu) \le f(y)$$

for any feasible pair (y, μ). The sign of equality holds iff (1)〜(n), $y = \mu$, (1)′〜(n)′ holds :

$$
\text{(EC)}\quad
\begin{aligned}
y_1 + y_2 - y_3 &= b_2\\
y_3 + y_4 - y_5 &= b_4\\
y_5 + y_6 - y_7 &= b_6\\
&\vdots\\
y_{2n-3} + y_{2n-2} - y_{2n-1} &= b_{2n-2}\\
y_{2n-1} + y_{2n} &= b_{2n}\\
\mu_1 - \mu_2 &= b_1\\
\mu_2 + \mu_3 - \mu_4 &= b_3\\
\mu_4 + \mu_5 - \mu_6 &= b_5\\
&\vdots\\
\mu_{2n-4} + \mu_{2n-3} - \mu_{2n-2} &= b_{2n-3}\\
\mu_{2n-2} + \mu_{2n-1} - \mu_{2n} &= b_{2n-1}\\
y_k &= \mu_k \qquad 1 \le k \le 2n.
\end{aligned}
$$

This is a system of $4n$ linear equations in $4n$ variables, which is equivalent to two (identical) systems of $2n$ equations in $2n$ variables. From Lemma 6, (EC) has a unique identical solution

$$(\hat{y}_1,\ \hat{y}_2,\ \ldots,\ \hat{y}_{2n}) = (\mu_1^*,\ \mu_2^*,\ \ldots,\ \mu_{2n}^*).$$

Lemma 6 (Fibonacci solution) *The system of* $2n$ *linear equations in* $2n$ *variables*

$$
\text{(NF)}\quad
\begin{aligned}
y_1 - y_2 &= b_1\\
y_2 + y_3 - y_4 &= b_3\\
y_4 + y_5 - y_6 &= b_5\\
&\vdots\\
y_{2n-4} + y_{2n-3} - y_{2n-2} &= b_{2n-3}\\
y_{2n-2} + y_{2n-1} - y_{2n} &= b_{2n-1}\\
y_1 + y_2 - y_3 &= b_2\\
y_3 + y_4 - y_5 &= b_4\\
y_5 + y_6 - y_7 &= b_6\\
&\vdots\\
y_{2n-3} + y_{2n-2} - y_{2n-1} &= b_{2n-2}\\
y_{2n-1} + y_{2n} &= b_{2n}
\end{aligned}
$$

has a unique solution:

$$
\begin{aligned}
y_1 &= \frac{1}{F_{2n+1}}(\,F_{2n}b_1 + F_{2n-1}b_2 + F_{2n-2}b_3 + \cdots\\
&\qquad + F_3b_{2n-2} + F_2b_{2n-1} + F_1b_{2n}\,)\\
y_2 &= \frac{1}{F_{2n+1}}(\,-F_{2n-1}b_1 + F_{2n-1}F_2b_2 + \cdots\\
&\qquad + F_3F_2b_{2n-2} + F_2F_2b_{2n-1} + F_2b_{2n}\,)\\
&\vdots\\
y_{2n-1} &= \frac{1}{F_{2n+1}}(\,F_2b_1 - F_2F_2b_2 + F_2F_3b_3 - \cdots\\
&\qquad - F_2F_{2n-2}b_{2n-2} + F_2F_{2n-1}b_{2n-1} + F_{2n-1}b_{2n}\,)\\
y_{2n} &= \frac{1}{F_{2n+1}}(\,-F_1b_1 + F_2b_2 - F_3b_3 + \cdots\\
&\qquad + F_{2n-2}b_{2n-2} - F_{2n-1}b_{2n-1} + F_{2n}b_{2n}\,).
\end{aligned}
$$

□

The equation (NF) is written as vector-matrix form

$$Ay = b$$

where A is the $2n \times 2n$ matrix, and y, b are the $2n$-vectors:

$$
A = \begin{pmatrix}
1 & -1 & 0 & \ldots & 0 & 0 & 0\\
1 & 1 & -1 & \ldots & 0 & 0 & 0\\
0 & 1 & 1 & \ldots & 0 & 0 & 0\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\
0 & 0 & 0 & \ldots & 1 & -1 & 0\\
0 & 0 & 0 & \ldots & 1 & 1 & -1\\
0 & 0 & 0 & \ldots & 0 & 1 & 1
\end{pmatrix},
$$

$$
y = \begin{pmatrix} y_1\\ y_2\\ \vdots\\ y_{2n} \end{pmatrix},\quad
b = \begin{pmatrix} b_1\\ b_2\\ \vdots\\ b_{2n} \end{pmatrix}.
$$

The matrix A has the inverse A^{-1} is

$$\frac{1}{F_{2n+1}}\begin{pmatrix} F_{2n} & F_{2n-1} & \cdots & F_2 & F_1 \\ -F_{2n-1} & F_{2n-1}F_2 & \cdots & F_2F_2 & F_2 \\ F_{2n-2} & -F_{2n-2}F_2 & \cdots & F_2F_3 & F_3 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -F_3 & F_3F_2 & \cdots & F_2F_{2n-2} & F_{2n-2} \\ F_2 & -F_2F_2 & \cdots & F_2F_{2n-1} & F_{2n-1} \\ -F_1 & F_2 & \cdots & -F_{2n-1} & F_{2n} \end{pmatrix}. \tag{14}$$

Let us define a $2n \times 2n$ diagonal matrix $J := \mathrm{diag}(a_{ii})$ with $a_{ii} = (-1)^i$:

$$J \;=\; \mathrm{diag}(-1, 1, -1, 1, \ldots, -1, 1).$$

Letting $B := JA^{-1}$, then we get

$$B \;=\; \frac{1}{F_{2n+1}}\begin{pmatrix} -F_{2n} & -F_{2n-1} & -F_{2n-2} & \cdots & -F_1 \\ -F_{2n-1} & F_{2n-1}F_2 & F_{2n-2}F_2 & \cdots & F_2 \\ -F_{2n-2} & F_{2n-2}F_2 & -F_{2n-2}F_3 & \cdots & -F_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -F_3 & F_3F_2 & -F_3F_3 & \cdots & F_{2n-2} \\ -F_2 & F_2F_2 & -F_2F_3 & \cdots & -F_{2n-1} \\ -F_1 & F_2 & -F_3 & \cdots & F_{2n} \end{pmatrix}.$$

Hence the equation (NF) has the solution mentioned above. Thus the primal (P) has a minimum value $m = (b, Bb)$ at the (minimum) point $\hat{y}$. The dual (D) has a maximum value $M = (b, Bb)$ at the (maximum) point μ^*. Thus both problems satisfy Fibonacci identical duality (FID).

A Homogeneous Pair

We consider a simple case $b_1 = b_2 = \cdots = b_7 = 0,\ b_8 = c$. Then $(\mathrm{P})''$ and $(\mathrm{D})''$ reduce to

$$(\mathrm{P})'' \quad \begin{array}{lrl} \text{minimize} & & y_1^2 + y_2^2 + \cdots + y_8^2 \\ \text{subject to} & \text{(i)} & y_1 + y_2 \;=\; y_3 \\ & \text{(ii)} & y_3 + y_4 \;=\; y_5 \\ & \text{(iii)} & y_5 + y_6 \;=\; y_7 \\ & \text{(iv)} & y_7 + y_8 \;=\; c \\ & \text{(v)} & y \in R^8 \end{array}$$

and

$$(\mathrm{D})'' \quad \begin{array}{lrl} \text{Maximize} & & -(\mu_1^2 + \mu_2^2 + \cdots + \mu_8^2) + 2c\mu_8 \\ \text{subject to} & \text{(i)}' & \mu_1 \;=\; \mu_2 \\ & \text{(ii)}' & \mu_2 + \mu_3 \;=\; \mu_4 \\ & \text{(iii)}' & \mu_4 + \mu_5 \;=\; \mu_6 \\ & \text{(iv)}' & \mu_6 + \mu_7 \;=\; \mu_8 \\ & \text{(v)}' & \mu \in R^8 \end{array}$$

, respectively. It turns out that $(\mathrm{P})''$ has a minimum value $m = \dfrac{F_8}{F_9}c^2$ and $(\mathrm{D})''$ has the *same* maximum value $M'' = \dfrac{F_8}{F_9}c^2$ at the *common* optimum point. Both problems have an *identical optimal solution* characterized by the first five Fibonacci numbers

$$F_1,\ F_2,\ \ldots,\ F_9.$$

Thus both the problems satisfy Fibonacci identical duality (FID):

1. (duality) $(\mathrm{P})''$ and $(\mathrm{D})''$ are dual to each other.
2. (identical) Both have an identical optimal solution (value and point).
3. (Fibonacci) $(\mathrm{P})''$ has a minimum value $m'' = \dfrac{F_8}{F_9}c^2$ at a minimum point

   $$\hat{y} = (\hat{y}_1,\ \hat{y}_2,\ \ldots,\ \hat{y}_8) \;=\; \frac{c}{F_9}(F_1,\ F_2,\ \ldots,\ F_8).$$

   $(\mathrm{D})''$ has a maximum value $M'' = \dfrac{F_8}{F_9}c^2$ at a maximum point

   $$\mu^* = (\mu_1^*,\ \mu_2^*,\ \ldots,\ \mu_8^*) \;=\; \frac{c}{F_9}(F_1,\ F_2,\ \ldots,\ F_8).$$

   Both the optimum points constitute a Fibonacci sequence.

References

E.F. Beckenbach and R.E. Bellman, *Inequalities*, Springer-Verlag, Ergebnisse 30, 1961.

R.E. Bellman, *Dynamic Programming*, Princeton Univ. Press, NJ, 1957.

R.E. Bellman, *Introduction to the Mathematical Theory of Control Processes, Vol.I, Linear Equations and Quadratic Criteria*, Academic Press, NY, 1967.

R.E. Bellman, *Methods of Nonlinear Analysis, Vol.I, Nonlinear Processes*, Academic Press, NY, 1969.

R.E. Bellman, *Introduction to the Mathematical Theory of Control Processes, Vol.II, Nonlinear Processes*, Academic Press, NY, 1971.

R.E. Bellman, *Methods of Nonlinear Analysis, Vol.II, Nonlinear Processes*, Academic Press, NY, 1972.

A. Beutelspacher and B. Petri, *Der Goldene Schnitt 2., überarbeitete und erweiterte Auflange*, ELSEVIER GmbH, Spectrum Akademischer Verlag, Heidelberg, 1996.

R.A. Dunlap, *The Golden Ratio and Fibonacci Numbers*, World Scientific Publishing Co.Pte.Ltd., 1977.

S. Iwamoto, *Theory of Dynamic Program*, Kyushu Univ. Press, Fukuoka, 1987 (*in Japanese*).

S. Iwamoto, *Mathematics for Optimization II Bellman Equation*, Chisen Shokan, Tokyo, 2013 (*in Japanese*).

S. Iwamoto and Y. Kimura, Nonhomogeneous Semi-Fibonacci Programming —Identical Duality—, RIMS Kokyuroku, Vol.2078, pp.114–120, 2018.

S. Iwamoto and Y. Kimura, Semi-Sibonacci Programming —from Sibonacci to Silver —, RIMS Kokyuroku, Vol.2126, pp.181–190, 2019.

S. Iwamoto and Y. Kimura, Semi-Fibonacci programming – odd-variable – , RIMS Kokyuroku, Vol.2158, pp.30–37, 2020.

S. Iwamoto, Y. Kimura and T. Fujita, On complementary duals – both fixed points – , Bull. Kyushu Inst. Tech, Pure Appl. Math., No.67, pp.1–28, 2020.

R.T. Rockafeller, *Conjugate Duality and Optimization*, SIAM, Philadelphia, 1974.

令和 3 年 7 月 30 日受付
令和 3 年 9 月 1 日受理

非同次セミフィボナッチ計画の双対

不等式アプローチ

木村寛[1]，岩本誠一[2]

[1] 秋田県立大学システム科学技術学部

[2] 九州大学名誉教授

ある最小化問題（P）と最大化問題（D）の 2 つの問題対を考える．これら（P）と（D）の目的関数は共に 2 次式であり制約は線形である．両問題の制約は非同次セミフィボナッチとよばれる．すなわち，これら問題は非同次制約をもつセミフィボナッチ計画とよばれる．本論文では主問題から双対問題を導出する双対化手法として不等式法による導出を提案する．この不等式法は相加相乗平均不等式を応用している．さらにこれらの最小化問題と最大化問題の対の間にはフィボナッチ一致双対性とよばれる性質が成り立つことを示す．フィボナッチ一致双対性は次の(i)〜(iii)の三位一体の関係を表す : (i) (Duality) 最小化問題（P）と最大化問題（D）は互いに双対である．(ii) (Identical) 最小化問題（P）と最大化問題（D）のそれぞれの最適点と最適値は共に一致する．(iii) (Fibonacci) 最小化問題（P）と最大化問題（D）の最適点と最適値は共にフィボナッチ数で表される．

キーワード : 最適化，2 次計画，フィボナッチ数列，双対化

責任著者連絡先：木村寛　〒015-0055 秋田県由利本荘市土谷字海老ノ口 84-4, 秋田県立大学システム科学技術経営システム工学科, E-mail: yutaka@akita-pu.ac.jp